Ordinal fair division under quotas

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July 2025

Abstract

A set A of m indivisible objects is to be fully allocated between n agents; each agent i is to get exactly q_i objects (so $\sum_i q_i = m$). Agents only report their ordinal preference orderings $\succeq_i$ of single objects in A. Thus, mechanism designer can only partially compare allocations of an agent, based on stochastic dominance.

We look for fair and efficient allocations, using new notions of fairness "up to one upgrade", stronger than traditional "up to one good" ones, and more appropriate for model with "quotas" q_i. Since individual shares' sizes q_i differ, fairness comparisons are based on the average valuation.

We demonstrate that in our model, ordinal efficiency ("OE", weaker condition than Pareto efficiency), and ordinal envy-freeness up to one upgrade ("oEFu1", stronger condition than cardinal one) are compatible.

We first show that the set of OE allocations is exactly the set of allocations obtained by queueing rules. Next, we show that a "fair" queue, which guarantees oEFu1 for any preferences, exists and is essentially unique.

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1 Introduction

This paper contributes to the now classical discrete fair division problem. A finite group of agents $N = \{1, ..., n\}$ needs to divide between themselves a finite set of objects $A = \{a_1, ..., a_m\}$ (items, goods/chores, tasks, resources, etc.). Objects are indivisible, and no monetary transfers or lotteries are available. Thus, the only feasible allocations are partitions $Z = (Z_1, ..., Z_n)$ of A. Agents have private valuations v_i for single objects, and additive utility functions over subsets of A ("shares", or "bundles") they might receive. The main concern of fair division literature is how to use heterogeneity of preferences to make all agents sufficiently happy. A first question would be: given a preference profile, does there exist an efficient and fair allocation? And if yes, how to find it? Efficiency is traditionally interpreted as simply Pareto Optimality (PO) (though we will depart from this – see below). The issue of fairness is much more subtle. Coming from continuous fair division models, two main notions of fairness are Proportionality (PROP) and Envy-Freeness (EF). Proportionality (also often called Fair Share) requires that each of n agents gets at least $\frac{1}{n}$-th of her valuation of the whole set A. Envy Freeness is a stronger condition, and demands that each agent views her share as the best one in the partition. In the discrete model, those properties are clearly out of reach due to indivisibilities. Think about A which consists of one diamond and many rocks. Anyone who does not get a diamond will envy the diamond owner, and will receive less then their fair share of A.

Thus, one can at best hope for some approximative fairness. An extensive literature on this model, which emerged in last two decades, uses fairness notions "up to one object", and dubs them PROP1/EF1. While an agent may value her share less then the appropriate fraction of the whole, or less then the share of another agent, the difference with the fair value does not exceed her valuation of a single object. Her share S would satisfy a property if she would be allowed to add one object to S or to disregard one object (in S, or outside S). This should be interpreted as a "thought experiment", rather then actual objects' transfer.

Operating with those definitions, preceding literature thoroughly investigated the existence of fair and efficient allocations, algorithms to compute those, their complexity, and many related questions. PROP1/EF1 are invariant with respect to linear transformations of utility, but heavily depend on the change of zeros. Thus, cases of "goods" (all $v_i \geq 0$), "bads"/"chores" ($v_i \leq 0$), or "mixed objects" are very different. A lot is known for the case of "goods". For example, an allocation which maximizes the product of utilities is both PO and EF1 (See Caragiannis at al. (2019) [7]). Cases of "chores" and, especially, arbitrary sign valuations, proved to be less tractable so far. See, in particular, Amanatidis at al. (2023) [1] for recent surveys on fair division of indivisible items.

We stay within the above discrete model, but impose an additional condition of exogenously specified shares' sizes q_i. In many practical instances, agents are indeed entitled to (or constrained by) pre-specified objective share sizes.

As a mock example, suppose that n agents order food together (pizzas, desserts, etc.). Each agent pre-specified the quantity she will want to eat. They receive a set of items in various flavors, each item is pre-cut in several slices, and further cutting is impractical. Agents have different tastes over flavors, they can mix and match (but not cut) slices, and each agent is to get in total exactly as many units as she ordered.

More seriously: Workers might need to divide a set of chores or shifts, while their contracts specify different numbers of working hours or projects to do. Team managers may want to split a given set of employees into teams of different specified sizes to complete different tasks. Dissolving a partnership or allocating an estate can come with different pre-specified "objectively" measured (for example, in terms of market monetary value) entitlements for involved parties. A processing server may need to schedule agents with portfolios of tasks of variable volumes. A charity might be distributing housing or food between family units or community groups of different sizes. Etc.

We assume that each agent i is assigned a size (or "quota") $q_i \in \mathbb{N}$, with $\sum_i q_i = m$. Her feasible potential shares are all subsets $S \subset A$ of the size $|S| = q_i$. While her utility from a feasible share S is still the sum of her valuations of objects in S, she might even not have a well defined utility for shares of other sizes (or this utility could be a large negative number). Hence, approximate fairness “up to 1 object” is a much less natural concept. We propose to require instead that an agent’s value from her share is fair up to a difference between utilities of two single objects (one from her bundle and one from outside). An important additional benefit of using approximations based on differences of utilities is that, combined with all feasible shares of a given agent being of the same size, it makes our setting invariant to affine transformations of utilities. Thus, we can treat cases of goods, chores or mixed objects simultaneously.

A companion paper (Bogomolnaia at al (2024) [5]) considers the case of identical quotas and adopts “up to one flip” (single objects’ exchange) notions, “PROPf1/EFf1”. Before verifying whether agent i’s share satisfies some fairness criterion, we are allowed to do a thought experiment of exchanging one of her objects for some object outside her share. PROPf1/EFf1 are neither stronger nor weaker than traditional PROP1/EF1.

We propose to use approximate fairness “up to 1 upgrade”, “PROPu1/EFu1”. Here, before checking fairness, an agent is allowed to “upgrade” one of her objects up to the best one from outside her share S (either from $A \setminus S$ or from the share T of another agent). Those concepts are stronger[1], and imply both PROP1/EF1 and PROPf1/EFf1. EFu1 is still stronger then PROPu1.

Importantly, in our model, given agents’ different size quotas, they cannot directly compare their shares with either shares of others, or with $\frac{1}{n}$-th of the total A. Instead, we adopt the notions of “fairness on average”. An agent i should believe the average “quality” (i.e., valuation) of objects in her share to be at least as good as the average of the whole set A (PROP), or at least as good as the average quality (in i’s own valuation) of objects in any other share (EF).

Consider for example, n project leads (agents) who are to divide a given set A of workers (objects) between them. Each lead i has to create a team to work on a specific project, and needs a specified number of workers q_i for it. They are looking for different skills sets, so their preferences over workers are different. A

[1] Arguably, those are the most strong fairness concepts feasible in discrete setting. See the discussion later.

“fair” allocation would be one where each lead thinks that the average quality of her workers is better than the average quality of workers in any other team, and/or than average quality of all workers.

Some of the preceding literature discussed various restrictions on the size of individual bundles. Most typical assumption is that objects come in different types and agents have some limits on number of objects from each type. Fairness is still evaluated based on total baskets' values. See, for example, the surveys by Suksompong (2021) [9], Biswas at al. (2023) [4]. A “weighted” model was also proposed, where agents are assumed to have different utility “entitlements” (without restrictions on bundle sizes), and fairness is evaluated in proportion to entitlements (see Aziz at al. (2020) [2], Chakraborty at al. (2021) [8]). This literature is concerned with the existence of traditional PROP1/EF1 fair allocations, and does not discuss efficiency.

While our fairness “on average” seems similar in spirit to that in weighted models, the implications are very different. We do not assume that some agents have more rights then others in terms of utilities. We assume instead that they have different exogenous restrictions. In particular, the sets of feasible allocations, and hence of efficient ones, are very different in those two models. Say, in our setting, a part time instructor has to teach 2 courses, while a full time one has to teach 4. We do not interpret it as if the full-time person has twice as much rights and so is entitled to twice as much utility. If this would be the requirement, it might well be that the best partition would assign different numbers of courses to each instructor – not 2 and 4, but 3 and 3 or 1 and 5. Instead, we aim for each instructor to like, on average, the courses she teaches better than the courses the other person is assigned to.

Our goal is to investigate till what extent efficiency and fairness are compatible in our model with size restrictions. A companion paper (Bogomolnaia at al. (2024) [5]), which only considers the case of equal shares ($q_i = q$ for all i), shows that the existence of PO and fair (either PROP1/EF1 or PROPf1/EFf1) allocations under quotas is a difficult open question so far. They are only shown to exist on some restricted domains, like identical utilities, or 0-1 utilities, or for two agents. We thus would like to investigate this problem from a very different prospective.

We depart from the notions of cardinal valuations reports and full Pareto Optimality, and work in the ordinal input framework.

We assume that agents have cardinal valuations for individual objects, and additive valuations of shares. However, agents only report their ordinal rankings of objects in A (which could be strict or not). An important attractive feature of this setting is its reduced information requirements. We ask much less from agents in terms of formulating and transmitting their preferences. Note also, that mechanisms observed in real life almost exclusively rely on ordinal data only.

This (incomplete) ordinal information still allows mechanism designer to compare some (but not all) potential shares from the point of view of a particular agent. Suppose that we know how an agent ranks individual objects, and she is presented with two bundles S and T of the same size. If she prefers her top object in S to her top object in T, her 2-nd best in S to her 2-nd best in T, etc., then we can be sure that she would value S above T, no matter what is her cardinal valuation function. We hence say that S "ordinally dominates" T for this agent. Otherwise, depending on her cardinal utility, her choice between S and T can go either way.

Ordinally Efficient (OE) partitions are those which are not ordinally dominated for all agents by any other partition. This is clearly a weaker notion then PO.

In the same spirit, we can define ordinal versions of PROP and EF, oPROP and oEF. Player i's share is oPROP/oEF iff it is PROP/EF for any valuation v_i compatible with her reported ranking $\succeq_i$. We further define their approximate variants, up to one upgrade etc. Contrary to OE, oPROP and oEF, as well as their approximations, are stronger requirements then cardinal ones.

Mechanisms with ordinal input, and corresponding ordinal notions of efficiency and fairness, were proposed and extensively studied in the framework of random assignment without money (see Bogomolnaia, Moulin (2001) [6]). As in our model, a finite set of indivisible objects is to be divided between agents with heterogenous preferences, and all agents have quotas on the number of objects. The standard case is n agents, n objects, and all quotas $q_1 = 1$. However, in an attempt to guarantee fairness (at least ex-ante), lotteries between deterministic allocations are allowed. While agents compare lotteries over objects based on their cardinal expected utilities, the mechanism only collects the information about agents' ordinal orderings of pure objects. We notice the conceptual parallel between comparing probability distributions over fixed number of objects and comparing baskets of objects of the same size, when only the ordering of single objects is known.

Our main result is to show that ordinal efficiency and ordinal fairness up to one upgrade, OE and oEFu1 (and hence oPROPu1), are always compatible. Moreover, there is essentially a unique method to guarantee OE and oEFu1 no matter agents' preferences.

In the case of identical quotas, i.e. when all $q_i = q$, it is relatively easy to show that good old Round Robin rule (when we order agents arbitrarily and let them choose single objects in turn, one by one, in $\frac{n}{q}$ rounds) fits the fairness bill. For arbitrary quotas, we first provide a complete characterization of OE allocations. They happen to be exactly the ones resulting from different queueing rules, where each agent i is present in the queue exactly q_i times, and each time it is her turn picks one object (the best for her among still available ones)[2]. We then show that among those queueing rules there always exists a "fair" one, which is an appropriate generalization of the traditional Round Robin queue, and that it guarantees our fairness requirements (oEFu1, and hence oPROPu1). Moreover, such an "EF balanced" queue is essentially unique[3].

Consider a queue $p = (p_1, ..., p_m)$ where each agent i appears q_i times. Intuitively, it is "fair" if all agents are evenly spread along p. More specifically, let $r^h[i]$ be the number of times agent i appears in a truncated queue $p^h = (p_1, ..., p_h)$. We would like, for each h, to have all fractions $r^h[i]/q_i$ (number of i-th positions in p^h relative to her total quota) to be approximately equal.

Queueing rules to pick objects in turn are closely related to "house monotone" apportionment methods of allocating parliament seats between n states with different populations q_i, when the number of house seats increases from 1 to $m = \sum q_i$. Our

[2] When indifferences are allowed, this definition is refined in an appropriate way to guarantee the full Pareto efficiency of the resulting queueing allocation if sub-agents are thought of as independent actors. See below.

[3] See the precise discussion later.

EF balanced queues correspond exactly to the Jefferson method of apportionment (see Balinski and Young (1981) [3] for a comprehensive discussion on apportionment rules).

In the context of weighted utilities model, Chakraborty at al. (2021) [8] also consider "fair" queues (which they call "pecking sequences"), in a similar to ours spirit. They do not look at efficiency, and use "up to 1 object" PROP1/EF1. Their notions of "weighted fairness" are based on entitlements to shares of welfare w_i, rather then quotas q_i on numbers of goods. Their "fair" queues approximately equalize $r^h[i]/w_i$ for each truncated queue p^h (which has different implications, compared to our model)[4]. They present a family of pecking sequences satisfying their fairness axioms, and connect them to different apportionment methods.

While a fair queue guarantees a fair allocation, we can of course have (ordinally) fair allocations resulting from very unfair queues. When preferences of agents are sufficiently heterogeneous, essentially any queue would generate a fair allocation. However:

Any OE allocation rule, for each preference profile $\succeq$, has to pick an allocation Z which results from some queue $p = p(\succeq)$. We might want this queue to be the same for all profiles $\succeq$ (say, we are deciding on an institutional policy of fair division, which will then apply in a variety of situations and for different sets of agents). We show that the only p, which guarantee fairness no matter $\succeq$, are "EF balanced" ones. Thus, there exists (essentially unique) institutional rule F_p which would guarantee OE and ordinal fairness for any preferences.

We would like to emphasize that our results are obtained for the full domain of cardinal/ordinal preferences, when agents potentially can be indifferent between objects. Full domain assumption is known to create many additional technical difficulties for mechanism design. Proofs become much more involved, results valid

[4] In weighted model, an agent typically does not get her exact fair share $w_i v_i(A)$ of total utility at the end of "pecking". In particular, if w_i are significantly different and $m = n$, some agents would get no objects, while some others would get at least two. Because of this, though in weighted model there are many different weighted "fair" queues, they do not include our "EF balanced" queue.

on the strict domain often fail to extend, various possibilities of tiebreaking allow for a wide variety of often intractable and unattractive rules to emerge, and many questions stay open. Many of our proofs in this paper are relatively straightforward for the case of strict preferences, compared to much more involved arguments we provide for the case with indifferences.

2 Notation and preliminaries: Ordinal Efficiency and Approximate Fairness

$N = \{1, ..., n\}$ is a set of agents, $A = \{a_1, ..., a_m\}$ is a set of objects, $|N| = n$, $|A| = m$. We will usually use letters i, j for agents and a, b, c, s, z for objects. Each $i \in N$ is entitled to a share of exactly $q_i \in \mathbb{N}$ objects, with $\sum_i q_i = m$, and has an ordinal ordering $\succeq_i$ over A. Presumably, it comes from some cardinal valuation function $v_i : A \to \mathbb{R}$ which respects this ordering, though v_i is not known to anyone except agent i.

We define the total valuation of a bundle $S \subset A$ to be $v_i(S) = \sum\limits_{a \in S} v_i(a)$.

We however assume that agent i's preferences over subsets of A (of any size!) are determined by the utility function based on average valuation: $u_i(S) = \frac{1}{|S|} \sum\limits_{a \in S} v_i(a)$.

A feasible allocation is a vector-partition of A into n sets, $Z = (Z_1, ..., Z_n)$, where $|Z_i| = q_i$ for all i, Z_i are disjoint, their union is A, and each agent i receives the share Z_i.

Given a set $S \subset A$, we will use notation $S = (s_1, ..., s_q)_i$ for a vector of elements of S, ordered in decreasing order of preferences of agent i (i.e., $s_1 \succeq_i ... \succeq_i s_q$).

Definition 1 Ordinal Dominance (OD) Let $S, T \subset A$, $|S| = |T| = q_i$, $S = (s_1, ..., s_{q_i})_i$, $T = (t_1, ..., t_{q_i})_i$. We say that agent i "ordinally prefers" S to T, and write $S \, OD_i \, T$, iff $s_k \succeq_i t_k$ for all k. OD_i is strict iff at least one of these inequalities is strict.

The binary relation OD_i is transitive but not complete one on the set of all $S \subset A$, $|S| = q_i$. The following equivalence will be important.

Lemma 1 *$S\,OD_i\,T$ if and only if for any cardinal valuation $v_i : A \to \mathbb{R}$ which respects $\succeq_i$ we have $\sum_{s\in S} v_i(s) \geq \sum_{t\in T} v_i(t)$ (strictly iff at least one of these inequalities is strict for at least one v_i).*

Proof. (of "$\Longleftarrow$", "$\Longrightarrow$" is obvious)

Assume $S = (s_1, ..., s_{q_i})_i$ does not ordinally dominate $T = (t_1, ..., t_{q_i})_i$ for agent i. Pick the smallest r such that $t_r \succ_i s_r$. Let $\varepsilon > 0 < \varepsilon < \frac{1}{2q_i}$. Consider a valuation v_i with $v_i(c) \geq 1 - \varepsilon$ for all $c \succeq_i t_r$ ("top objects"), and $v_i(d) \leq \varepsilon$ for all d with $t_r \succ_i d$ ("bottom objects"). Then $v_i(T) \geq r(1-\varepsilon) \geq r - q_i\varepsilon$, while $v_i(S) \leq (r-1) + (n-r+1)\varepsilon \leq r - 1 + q_i\varepsilon$. Hence, $v_i(T) \geq v_i(S)$. ■

When mechanism designer only knows ordinal input, the only way for him to guarantee that an agent prefers given bundle S to another T is to make sure that S ordinally dominates T for this agent. In terms of efficiency, usually the best ordinal design can guarantee is that the proposed allocation would not be dominated. We thus define

Definition 2 Ordinal Efficiency (OE) Let $Z = (Z_1, ..., Z_n)$ and $Y = (Y_1, ..., Y_n)$ be two feasible allocations. We say that Z ordinally dominates Y, and write $Z\,OD\,Y$, iff $Z_i\,OD_i\,Y_i$ for all $i \in N$, and domination is strict for at least one i. Ordinally Efficient (OE) partitions are those which are not strictly dominated by any other partition.

Ordinal Efficiency is clearly a weaker concept then full Pareto Optimality. We however, still interpret fairness as evaluated by an agent herself (and she is aware of her cardinal valuations). Consistently with the previous literature on ordinal input mechanism design, the notions of fairness we consider (Proportionality and Envy-Freeness) can be re-defined in an ordinal way. Relying on Lemma 1, we define an "ordinally fair" allocation as one which is "fair" for ALL cardinal valuations which are compatible with the reported ordinal orderings. Those are very strong requirements. They guarantee corresponding "fairness" in full, even though mechanism designer has incomplete information on agents' preferences.

Indeed, efficiency is evaluated for the whole allocation, and by the mechanism designer, who has incomplete information about agents' preferences (just their ordinal orderings), so full PO is usually out of reach. However, fairness is evaluated by each agent herself, and she knows her own preferences in full. In order to guarantee fairness, the designer has to pick an allocation which would be fair for all potential cardinal valuations.

When defining approximate fairness in our model, we keep in mind that, for each agent i, only shares of size q_i are feasible. As in traditional discrete fair share setting, we allow for a small change in allocation to be contemplated before a fairness criterion is checked. However, instead of disregarding an object or adding it to i's allocation, we will allow an agent to "upgrade" one of her goods to the value of the best for her outside object. This will keep her allocation feasible.

We also need to clarify how an agent compares her share with a share of different size (so not feasible for her), belonging to someone else, or with the total A. We submit that a natural way of comparison in this case would be by "average value" of the objects. When comparing her bundle with another bundle of probably significantly different size, an agent cares about the average "value" or "quality" of both, not about sums of the valuations of objects in each bundle. If, say, a part-time worker, who by her contract only works 10 hours per week, compares her set of tasks with that of a full-time person, she deems an allocation of projects fair if in her opinion the average value (difficulty, level of interest, etc.) of her assigned projects is at least as good as the average value of the projects assigned to that other person.

Our fairness notions are proportionality and envy freeness "up to one upgrade". We refer to them as PROPu1 and EFu1. Before comparing her share with that of another agent or with the whole set A, an agent i is allowed to do a "thought experiment" of upgrading one item in her bundle to the value of an outside item (from the bundle of that other agent or from the whole A). Our approximate fairness is ensured if post upgrade i would think the allocation fair.

Let $S, T \subset A$, $a, b \in A$. We will often use notation "$S + T$", "$S + a$" for $S \cup T$, $S \cup \{a\}$ (especially when S, T are disjoint, or $a \notin S$), and "$S - T$", "$S - a$" for $S \backslash T$, $S \backslash \{a\}$ (especially when $T \subset S$, $a \in S$).

For any $S \subset A$, any $a \in S$, and any $b \notin S$, we define $S^{ba} = S \cup \{b\} \backslash \{a\} = S + b - a$, and refer to S^{ba} as "S after b-a flip" or "S after b-a upgrade". This is a set agent

i can obtain if she is allowed to "upgrade" her share S by substituting one of her objects a by some object b outside her bundle.

Fix a feasible allocation $Z = (Z_1, ..., Z_n)$, and an agent i with ordinal ranking $\succeq_i$ over A; assume she also has a cardinal valuation function $v_i : A \to \mathbb{R}$ representing this ranking. Recall that $u_i(S) = \frac{1}{|S|} \sum_{s \in S} v_i(s)$.

Definition 3 Fairness (Proportionality principle)
– Agent i is PROP at Z iff $u_i(Z_i) \geq u_i(A)$.
– Agent i is PROPu1 at Z iff either she is PROP or there exists Z_i^{ba}, an b-a upgrade of Z_i, such that $u_i(Z_i^{ba}) \geq u_i(A)$.
– Agent i is PROP'u1 at Z iff either she is PROP or there exists Z_i^{ba}, an b-a upgrade of Z_i, such that $u_i(Z_i^{ba}) \geq u_i(A \backslash Z_i)$.
– Agent i is oPROP• at Z iff she is PROP• for any v_i respecting $\succeq_i$.

Under PROPu1, an agent compares her upgraded share with the whole set A. Under PROP'u1, she compares it with the the set of all objects other agents get. While we believe that PROPu1 is more appropriate then PROP'u1, we introduce both for completeness. Also, as we will see, as PROP'u1 is an intermediate property between PROPu1 and EFu1, using it allows us to simplify some of our arguments.

Definition 4 Fairness (No Envy principle)
– Agent i is EF (envy-free) w.r.t. agent j at Z iff $u_i(Z_i) \geq u_i(Z_j)$.
– Agent i is EFu1 w.r.t. agent j at Z iff either she is EF w.r.t. j or there exists an b-a upgrade of Z_i with $b \in Z_j$, such that $u_i(Z_i^{ba}) \geq u_i(Z_j)$.
– Agent i is oEF• w.r.t. agent j at Z iff she is EF• w.r.t. j for any v_i respecting $\succeq_i$.
– Agent i is EF•/oEF• iff she satisfies this requirement w.r.t. all $j \neq i$.

We submit that our definitions are not just the most appropriate for our setting, but also the most strong requirements among those known.

Let us compare fairness up to one upgrade to other notions of approximate fairness, "up to one object" (PROP1/EF1) and "up to one flip" (PROPf1/EFf1). Both of them were mostly used for the case where agents are interested in the total valuations of bundles $v_i(S) = \sum_{a \in S} v_i(a)$, not the averages $u_i(\cdot)$.

Under “up to one object” relaxation, definitions (and results) differ depending on the signs of v_i. They require that an individual share Z_i should become fair if we allow an agent i to either add one object to Z_i (an appropriate action for “goods”, when $v_i(\cdot) \geq 0$), or to disregard one of her objects in Z_i (an appropriate action for “chores”, when $v_i(\cdot) \leq 0$). The underlying idea is that, though agent i's utility could be smaller than her fair value, the difference between her valuations of her own bundle and of the set she compares it with is small, in that it does not exceed her value of a single object.

Various extensions and relaxations of the “up to one good” principle were proposed for the case of mixed objects. Our goal is not a full comparison of all possible variants, so we concentrate on the cases when all $v_i(\cdot)$ are of the same sign.

Fairness “up to one object”

– Agent i is PROP1 at Z (under v_i) iff either she is PROP or there is either $b \notin S_i$ such that $u_i(Z_i + b) \geq u_i(A)$, or $a \in Z_i$ such that $u_i(Z_i - a) \geq u_i(A)$.

– Agent i is EF1 w.r.t. agent j at Z (under v_i) iff either she is EF w.r.t. j or there exists either $b \in Z_j$ such that $u_i(Z_i + b) \geq u_i(Z_j)$, or $a \in Z_i$ such that $u_i(Z_i - a) \geq u_i(Z_j)$.

– Here, EF1 implies PROP1.

“Up to one flip” relaxation was introduced for the case with identical quotas as a more appropriate one when feasible size of an individual bundle is fixed. Note that when all $q_i = q$ there is no difference between evaluating bundles according to either totals or averages.

Up to one flip properties do not depend on the sign of v_i. They require that an individual share Z_i should become fair if we allow an agent i to do one “flip”, exchanging one of her objects for one object outside her bundle. For the case of EFf1, contrary to “up to one upgrade” notion, a flip changes not only the share of this agent i, but also the share Z_j of the other agent j with which she compares her Z_i. A flip means that j gives one of his objects to i and gets her object in return.

Here, again, though agent i's utility could be smaller than her fair value, the difference between her valuations of her own bundle and of the set she compares it with is small. But now “small” means “not more then twice difference between utilities of two objects".

Fairness "up to one flip"

– PROPf1 is the same as PROPu1.

– Agent i is EFf1 w.r.t. agent j at Z (under v_i) iff either she is EF w.r.t. j or there exists an b-a flip of $a \in Z_i$ with $b \in Z_j$ such that $u_i(Z_i^{ba}) \geq u_i(Z_j^{ab})$.

– EFf1 does not imply PROf1, but guarantees $\frac{1}{2}$-PROPf1 (see Bogomolnaia at al. (2024) [5]).

There is an important conceptual difference between traditional PROP1/EF1 properties and "up to one flip"/"up to one upgrade" ones.

For simplicity of notation, consider the case of identical quotas $q_i \equiv q$. In this case, we can use comparisons based on $v_i(S) = \sum_{b \in S} v_i(b)$ for all fairness properties.

EF1 can be rephrased as: $v_i(Z_j) - v_i(Z_i) \leq \max_{b \in Z_j} v_i(b)$ (case of "goods") or $v_i(Z_j) - v_i(Z_i) \leq -\min_{a \in Z_i} v_i(a)$ (case of "chores"), i.e. "even if i prefers j-th bundle, she believes that the utility difference between two bundles is small — no more then her valuation of a single object".

EFf1 (up to one flip) requires $v_i(Z_j) - v_i(Z_i) \leq 2[\max_{b \in Z_j} v_i(b) - \min_{a \in Z_i} v_i(a)]$, "the difference of utilities between my and your bundles is at most twice the difference between single objects' valuations - one yours and one mine".

Finally, for our EFu1 (up to one upgrade), the requirement is
$v_i(Z_j) - v_i(Z_i) \leq \max_{b \in Z_j} v_i(b) - \min_{a \in Z_i} v_i(a)$.

More generally, we could consider, for any $K > 0$, a requirement "EF[K]":
$v_i(Z_j) - v_i(Z_i) \leq K[\max_{b \in Z_j} v_i(b) - \min_{a \in Z_i} v_i(a)]$,
so EFf1 is EF[2], and EFu1 is EF[1]. In the example with one diamond and many rocks, however, there are no EF[K] allocations for $K < 1$. Thus, our EFu1 is probably the strongest meaningful requirement for approximate envy freeness.

It is easy to see from the above discussion[5] that

Lemma 2 *(1) EFu1 implies PROP'u1 (and they are equivalent for $n = 2$).*

[5] See also the discussion about PROP1/EF1 versus PROPf1/EFf1 in Bogomolnaia at al. (2024) [5].

(2) EFu1 is stronger then both EF1 and EFf1 (who do not imply each other).

(3) PROP'u1 is stronger than PROPu1. PROPu1 implies PROP1 (for same sign valuations) and is the same as PROPf1.

Proof. (of (1), the rest follows from the discussion above)

Let $Z = (Z_1, ..., Z_n)$ be EFu1 for agent i. Let $a \in \arg\min_{c \in Z_i} v_i(c)$, $b \in \arg\max_{c \in A - Z_i} v_i(c)$, $b \in Z_k$ for some $k \neq i$. Thus, Z_i^{ba} is (one of) the best possible upgrade(s) for agent i. Let $S \in \arg\max\{u_i(Z_i), u_i(Z_i^{ba})\}$. By EFu1, $U' = u_i(S) \geq u_i(Z_j) = U_j$ for all $j \neq i$. Hence, $U' \geq \frac{1}{m-q_i} \sum_{j \neq i} q_j U_j = u_i(A \backslash Z_i)$ and we have PROP'u1. ■

Our notions of approximate fairness "up to one upgrade" (as well as "up to 1 flip" ones) are based on differences between utilities of two objects, not on a valuation of a single object. In addition, in our setting all potential feasible bundles of a given agent have the same size.

As a result, these fairness properties are invariant under affine transformations of utilities. If we rescale valuations of an agent i from v_i to $\widehat{v}_i(\cdot) = \alpha_i v_i(\cdot) + \beta_i$, with any $\alpha_i > 0$ and any β_i, her preferences over feasible bundles do not change, as well as whether a given allocation is PROPu1, PROP'u1 or EFu1 for her. However, it can affect PROP1 and/or EF1 when $\beta_i \neq 0$. Those traditional properties are preserved under linear transformations, but are sensitive to the changes of zeros. As an illustration, let $Z_1 = \{a, a, a\}$, $Z_2 = \{b, b, b\}$, and the valuations change from $v_1(a) = 1$, $v_1(b) = 4$, to $\widehat{v}_1(\cdot) = v_1(\cdot) + 10$. Under v_1, Z is neither PROP1 not EF1 for agent 1, while under $\widehat{v}_1(\cdot)$ the same Z satisfies both.

In particular, when we investigate compatibility of efficiency and fairness "up to one upgrade", we can normalize utilities in various convenient ways, and treat simultaneously the cases of "goods", "chores", or mixed items. This applies to both ordinal and cardinal setting. If we are looking for fairness "up to one good", those cases are very different even under quotas (as is it is in the unrestricted model).

We will now discuss a natural set of allocation rules in our model, based on agents queueing for objects. It turns out that those rules generate exactly the whole set of ordinally efficient allocations. Moreover, as we will see, there always exist "approximately fair" ways to organize the queue, which guarantee the (stronger) ordinal versions of PROPu1, PROP'u1, and EFu1 properties.

<u>Queueing rules</u>

Fix N, A, $q = (q_1, ..., q_n)$, and $\succeq = (\succeq_1, ..., \succeq_n)$.

For each agent i, create q_i sub-agents, $i^1, ..., i^{q_i}$, who all have the same preference ordering $\succeq_i$ and are entitled to one object each.

Let $Z = (Z_1, ..., Z_n)$ be a feasible allocation. Assign each Z_i to q_i sub-agents of agent i in an arbitrary way (each sub-agent gets one object from S_i). We treat the resulting allocation $\widetilde{Z}$ as a feasible allocation to $m = \sum_i q_i$ sub-agents, thought of as independent agents, each entitled with size quota 1. It is easy to see the following

Lemma 3 *Allocation Z is OE for the set N of n agents $1, ..., n$ if and only if any corresponding $\widetilde{Z}$ is PO as a feasible allocation to those m sub-agents, each entitled to one good.*

Proof. "$\Longrightarrow$" Consider an OE Z. Suppose that for a corresponding allocation to sub-agents $\widetilde{Z}$ there exists a Pareto improvement $\widetilde{Y}$, and let $k = i^r$ be a sub-agent of i who strictly prefers $\widetilde{Y}$ to $\widetilde{Z}$. Then in corresponding to $\widetilde{Y}$ allocation to agents $Y = (Y_1, ..., Y_n)$ we have $Y_j \, OD_j \, Z_j$ with strict domination for agent i. Hence Y ordinally dominates Z.

"$\Longleftarrow$" Let $\widetilde{Z}$ be a PO for sub-agents, corresponding to $Z = (Z_1, ..., Z_n)$. Suppose that $Y = (Y_1, ..., Y_n)$ ordinally dominates Z with $Y_i \, OD_i \, Z_i$ being strict for some agent i. We can then construct $\widetilde{Y}$ corresponding to Y, where each sub-agent j^r is assigned a good y^i_r which she prefers to her good in $\widetilde{Z}$, with at least one preference (for one of i-th sub-agents) being strict. This $\widetilde{Y}$ will Pareto dominate $\widetilde{Z}$. ■

A queue p is an ordering of all m sub-agents; $p = (p_1, ..., p_m)$.

Define $p^h = (p_1, ..., p_h)$ to be the truncated at h queue p (one consisting of its first h elements). Further, for any queue p and $h = 1, ..., m$, let $r^h[i] = r^h(p)[i] \in \mathbb{Z}_+$ be the number of sub-agents of i who appear in the truncated queue p^h.

<u>Queueing allocations</u>

Corresponding to p (feasible) queueing allocation $\pi(p)$ is obtained as follows.

Let Ω_0 be the set of all feasible allocations $Z = (Z_1, ..., Z_n)$ for the set N of agents, where each share $Z_i = (z^i_1, ..., z^i_{q_i})_i$ is ordered in decreasing order of preferences of agent i, and it is assumed that the sub-agents of i will be allocated objects from z^i_1 to $z^i_{q_i}$ in the same order in which they appear in the queue p.

For any $h = 1, ..., m$, define $\Omega_h \subset \Omega_{h-1}$ to be the set of all best for agent p_h allocations in Ω_h, (those which give the sub-agent p_h one of her best possible goods).

Finally, $\pi(p) = \Omega_m$.

Note: When all preferences are strict, this is simply the unique allocation where each sub-agent, when it is her turn, picks the best for her object among still available ones. When indifferences are present, the algorithm in the definition above could return not a single allocation, but a set of feasible allocations. However, each agent (and each sub-agent) will be indifferent between all allocations in $\pi(p)$, so it is always a singleton preferences-wise.

Given a queue p and an arbitrary feasible allocation $Z = (Z_1, ..., Z_n)$, with all Z_i ordered as above, $Z_i = (z^i_1, ..., z^i_{q_i})_i$ (Z is not necessarily in $\pi(p)$ or even OE!), we may arrange the objects in A in the order they should be picked by sub-agents in p so as to obtain Z. We denote this ordering by $A(p, Z) = (b_1, ..., b_m)$. Here sub-agents do not choose their best objects but rather those "prescribed" for them by Z. Specifically, let p_h be the l-th sub-agent of agent j in the queue p. She then has to pick $b_h = z^j_l \in Z_j = (z^j_1, ..., z^j_{q_j})_j$.

If $Z \in \pi(p)$ and all vectors $Z_i = (z^i_1, ..., z^i_{q_i})_i$ arrange objects in exactly the same order as the one in which they are picked to obtain Z by queueing algorithm (this is important!), then $A(p, Z)$ is exactly the same order in which objects are picked under the queuing algorithm.

3 Results

We start by establishing the equivalence between the set of all OE allocations and the set of all allocations obtained by different queueing rules. By Lemma 3, the following two statements are equivalent. We will prove the second one. For better understanding, we first present the proof for the case when orderings are strict, as it is standard and relatively straightforward. When indifferences are allowed, the argument is much more involved.

Theorem 1 *A feasible allocation Z is OE if and only if there exists a queue p such that $Z \in \pi(p)$.*

Theorem 2 *A feasible allocation Z of m goods to m agents, each of whom is entitled to one good, is PO if and only if there exists a queue p such that $Z \in \pi(p)$.*

Proof. (of Theorem 2)

"$\Longleftarrow$" Suppose that Z, resulting from some $p = (p_1, ..., p_m)$, is not PO. Then there is Z' which is better for all agents (weakly for all, and strictly for at least one). Let $(a_1, ..., a_m)_Z$ be the order at which objects in A are picked using the queue p (resulting in our allocation Z). Thus, agent p_k gets object a_k. Let $(a'_1, ..., a'_m)_{Z'}$ be the ordering of A, obtained by asking agents to pick the objects allocated to them by Z' in turn, according to p. Here agent p_k gets object a'_k. Let h be the first moment in the queue when $a_h \nsim a'_h$. Since Z' Pareto dominates Z, such a moment exists, and it has to be $a'_h \succ_{i_h} a_h$. But this contradicts the definition of the queueing rule (as agent h could get a better than a_h object without harming agents in front of her in the queue).

"$\Longrightarrow$" (*For strict preferences*) Fix a PO allocation Z, where each agent i gets an object z_i.

First, let each agent point at her (unique) best object in A, and let each object point at the agent to whom it is assigned. If no agent would point at her own object, then there would be a cycle. By assigning to each agent in this cycle the object to which she points we would obtain a Pareto improvement for Z. Thus, there is an agent who gets her best object at Z. Label her p_1. Remove both agent p_1 and the object a_1 she gets in Z, and repeat the same argument for $N - p_1$ and $A - a_1$. It gives us agent p_2, who gets her best object in $A - a_1$; etc. After m steps we obtain a queue $p = (p_1, ..., p_m)$, with $Z = \pi(p)$.

"$\Longrightarrow$" (*For domain with indifferences*) Fix a PO allocation Z, where each agent i gets an object z_i. We will construct a queue p such that $Z \in \pi(p)$.

Step 1. In any PO Z, there is an agent who is getting one of her best objects. Indeed, if not, let each agent i point at one of her best objects (which are all strictly better for her then z_i), and let each object a point at the agent to whom it is assigned (i.e., at i such that $a = z_i$). Since there is an arrow coming out from each agent and from each object, there will be a cycle. If we exchange items between agents in this cycle, by giving each agent the object at which she points, we obtain Pareto improvement.

Pick one of agents who gets her best object, and place her into the first position p_1 in the queue p. Obviously, $Z \in \Omega_1 = \Omega_1(p^1)$, the set of allocations which give this agent i one of her best objects.

Step h. Assume that a truncated sub-queue of agents p^{h-1} is already chosen, and $Z \in \Omega_{h-1} = \Omega_{h-1}(p^{h-1})$. Let $Q = \{p_1.,,,p_h\} \subset N$ and $F = N - Q$ ("free agents").

We will show that there exists an agent $j \in F$, such that the object she gets in Z is (one of) the best she can get, provided that each agent $k \in Q$ gets something at least as good for this k as z_k. We then will choose one of such agents j as next agent p_h in our queue p. This would guarantee $Z \in \Omega_h = \Omega_h(p^h)$.

By contradiction, assume that there is no such agent $j \in F$. We will construct the following bipartite oriented graph with $2m$ vertices, corresponding to m objects and m agents.

– Let each object point to the agent to whom it is allocated under Z (so different objects point to different agents!).

– Let each agent $k \in Q$ point to all objects, equivalent for her to z_k, the object she gets at Z.

– Let each agent $j \in F$ point to all the best objects she can get, provided that any agent $k \in Q$ gets an object at least as good for this agent k as z_k. Note that under our assumption all objects to which j points are strictly better for her then z_j.

Next, we construct a path along the arrows, j_1, a_1, j_2, a_2, j_3, a_3,... in the following way. Start from an arbitrary agent $j_1 \in F$.

– For any object a_s on our path pick j_{s+1} to be the unique agent at which it points (the agent who gets this object under Z).

– For any agent $j_s \in F$ on our path, pick as a_s any of the objects to which she points.

– Let $j_s \in Q$ (hence $s > 1$), while $j_{s-1} \in F$. Agent j_{s-1} points on our path to the object a_{s-1} she prefers to $z_{j_{s-1}}$. By the definition of our graph, there exists an allocation $Y = Y(j_{s-1})$, such that this agent j_{s-1} gets in Y the object $y_{j_{s-1}} = a_{s-1}$, while every agent $k \in Q$ gets in Y some object $y_k \sim z_k$. Fix such Y, and continue picking $a_l = y_{j_i}$ for agents j_l on our path, as long as $j_l \in Q$ (i.e., until we encounter an agent from F).

Since our graph is finite, there will be a repeated vertex on this path, and hence a cycle. If the first repeated vertex is an agent, $j_{r_1} = j_{r_2}$, $r_1 < r_2$, then this agent is $j_{r_1} = j_1 \in F$ (otherwise two different objects, a_{r_1-1} and a_{r_2-1}, would point to the same agent).

If the first repeated vertex is an object, $a_{r_1} = a_{r_2}$, $r_1 < r_2$, then we obtain a cycle a_{r_1}, j_{r_1+1}, a_{r_1+1},...,j_{r_2}, $a_{r_2} = a_{r_1}$. If all agents in this cycle are from Q, then there is $s^* \leq r_1$ such that $j_{s^*} \in F$, while j_{s^*+1},..., j_{r_1}, j_{r_1+1},..., j_{r_2-1}, $j_{r_2} = j_{r_1} \in Q$. Hence, there exists an allocation $Y = Y(j_{s^*})$ such that $a_k = y_{j_k}$ for all $k \in \{j_{s^*}, ..., j_{r_2}\}$. In particular, on our path, $y_{j_{r_1}} = a_{r_1} = a_{r_2} = y_{j_{r_2}}$, which contradicts the fact that at Y different agents get different objects.

Thus, at least one agent in our cycle is from F, and so she points at an object in this cycle, which is strictly better for her then her object in Z. But then, we can exchange items between agents in this cycle, by giving each agent the object at which she points, and we obtain Pareto improvement, the desired contradiction.

After m steps we construct a full queue p. It is easy to see that $Z \in \Omega_n = \Omega_n(p)$, one of the (utility equivalent) allocations resulting from this queue. ■

We now turn to discuss fairness. We already know that to guarantee efficiency we must allocate objects by some queueing rule. Thus, we need to understand which queues result in most fair outcomes.

We will first introduce the notions of “fair”, or “balanced”, queues. Those are queues p, where each agent approximately evenly spreads her sub-agents along the queue, and no one is much ahead of someone else.

We will then show that balanced queues produce “fair” (up to one upgrade) outcomes. Moreover, if one wants to use the same queue for all possible preferences, and needs the outcome to be fair, one has to pick a balanced queue.

Definition 5

(i) A queue p is PROP balanced for agent i, iff any truncated queue p^h contains at least $\frac{q_i}{m}h-1$ sub-agents of i: $r^h(p)[i]+1 \geq \frac{q_i}{m}h$.

(i) A queue p is PROP' balanced for agent i, iff in any truncated queue p^h we have $\frac{1}{q_i}\left(r^h(p)[i]+1\right) \geq \frac{1}{m-q_i}(h-r^h(p)[i])$, or, equivalently, $r^h(p)[i]+1 \geq \frac{q_i}{m}(h+1)$.

(iii) A queue p is EF balanced for agent i (with quota q_i) w.r.t. agent j (with quota q_j), iff in any truncated queue p^h we have $\frac{1}{q_i}\left(r^h(p)[i]+1\right) \geq \frac{1}{q_j}r^h(p)[j]$.

EF balancedness requires that, at any step h in the queue, if one would add one sub-agent to the total number of representatives of i in the truncated queue p^h, then the fraction of sub-agents of j in p^h, related to the total quota of j, will not exceed the fraction of sub-agents of i in p^h, related to the total quota of i. The following is very easy to see[6].

Lemma 4 *(1) A queue p is EF balanced for agent i w.r.t agent j, iff its sub-queue p^{ij}, for q_i+q_j objects and 2 agents, obtained by deleting from p all sub-agents except those of i and j, is EF balanced for agent i.*

(2) For $n=2$, EF balancedness for i is equivalent to PROP' balancedness for i. For $n>2$, EF balancedness is stronger.

(3) PROP' balancedness implies PROP balancedness (inverse is not true even for $n=2$).

We will show that, for any vector of quotas, there always exist such balanced “fair queues”. Further, we will check that our “fair queues” guarantee our corresponding approximate notions of fairness.

[6] Using the fact that, for any $x, y_1, ..., y_k \geq 0$, if $x \geq y_j$ for all j, then $x \geq \sum \lambda_j y_j$ for any $\lambda_j \geq 0$ with $\sum \lambda_j = 1$.

Existence follows from an explicit characterization of EF balanced queues, provided below. Those are queues p, for which, at any position h, an agent i is chosen to be p_h in the following way. We look at the truncated queue p^{h-1}, and calculate for each agent j what would be her number of sub agents in p^h, relative to her quota q_i, if we would add her next. I.e. we calculate $\frac{1}{q_j}\left(r^{h-1}[j]+1\right)$ for all agents j. Agent $i = p_h$ has to be one for whom this expression is the smallest.

Theorem 3 *Fix a set N of agents with quotas $q_i \in \mathbb{Z}_+$, where $\sum_{i=1}^{n} q_i = m$. The following two statements are equivalent. Let $p = (p_1, ..., p_m)$ be some queue of agents (with repetitions).*

(1) This queue $p = (p_1, ..., p_m)$ satisfies quotas (each agent i appears there q_i times) and is EF balanced (for any agent and with respect to any agent).

(2) For any $h = 1, ..., m$ we have $p_h \in \arg\min_{i \in N} \frac{1}{q_i}\left(r^{h-1}[i]+1\right)$ (here $r^{h-1}[i] = r^{h-1}(p)[i]$ is the number of times agent i appears in the truncated queue $p^{h-1} = (p_1, ..., p_{h-1})$).

Proof. *(1) $\Rightarrow$ (2)* By contradiction, suppose h is the smallest position in the queue p, such that $\frac{1}{q_i}\left(r^{h-1}[i]+1\right) > \frac{1}{q_j}\left(r^{h-1}[j]+1\right)$ for agent $i = p_h$ and some agent $j \neq i$. Then at the position h we have $\frac{1}{q_j}\left(r^{h}[j]+1\right) = \frac{1}{q_j}\left(r^{h-1}[j]+1\right) < \frac{1}{q_i}\left(r^{h-1}[i]+1\right) = \frac{1}{q_i}\left(r^{h}[i]\right)$, so our queue p is not EF balanced for agent j with respect to agent i.

(2) $\Rightarrow$ (1) Let p be constructed sequentially, by choosing an agent p_h for each next position h to be (one of) the agents for which $\frac{1}{q_i}\left(r^{h-1}[i]+1\right)$ is the smallest.

First, no agent will get more then her quota positions in p, thus, once the whole queue is constructed, each agent i appears exactly q_i times.

Indeed, if $r^{h-1}[i] = q_i$, then $\frac{1}{q_i}\left(r^{h-1}[i]+1\right) > 1$, while for all agents j with $r^{h-1}[j] < q_j$ we have $\frac{1}{q_j}\left(r^{h-1}[j]+1\right) \leq 1$. At least one such agent j exists for any $h \leq m$, since $\sum_i q_i = m > h-1$. Thus, agent i will not be chosen as p_h.

Second, we show EF balancedness by induction in h.

For $h = 1$, EF balancedness reduces to $\frac{1}{q_j} \geq \frac{1}{q_{p_1}}$ for all $j \neq p_1$, which is true by the choice of p_1.

Let $h \geq 2$, and $p_h = i$. Since EF balancedness inequality was true up to $h-1$, we only need to check it at the position h, and for agents $j \neq i$ with respect to agent

i, the one for whom $r^{\cdot}(\cdot)$ changes: $r^h(i) = r^{h-1}(i) + 1$. But given the way $i = p^h$ is chosen, we have $\frac{1}{q_i} r^h[i] = \frac{1}{q_i} r\left({}^{h-1}[i] + 1\right) \leq \frac{1}{q_j}\left(r^{h-1}[j] + 1\right)$. ■

Theorem 3 tells us that the EF balanced p are exactly those where at each step h agent p_h is one of i maximizing $\frac{1}{q_i}\left(r^{h-1}[i] + 1\right)$.

Thus, the last n positions $(p_{m-n+1}, ...p_m)$ in any EF balanced queue p will always contain one sub-agent for each agent, and they can be ordered in any way. Indeed, the last sub-agent of i may only appear in the queue after all other agents j got at least $q_j - 1$ positions each.

When all fractions $\frac{k_i}{q_i}$, $i \in N$, $0 < k_i < q_i$, are different, an EF balanced queue is unique up to p^{m-n}. Otherwise, we obtain a small well-defined family of fair queues[7]. In particular:

Corollary 3.1 (the case of equal quotas)

Let all $q_i \equiv q$ and $m = nq$. A queue p is a (satisfying quotas) EF balanced queue, if and only if each its sub-queue $(p_{nk+1}, ...p_{(k+1)n})$, for $k = 0, 1, ..., q-1$, contains all n agents.

Rephrasing, EF balanced queues for $q_i \equiv q$ are those which proceed in q rounds, and in each round every agent has exactly one turn to choose an object (though in different rounds agents may be ordered differently). We dub such queues "generalized Round Robin" (gRR) ones.

Proof. (of Corollary 3.1)

By Theorem 3, under identical quotas, EF balancedness is equivalent to the condition $p_h \in \arg\min_{i \in N} \frac{1}{q_i}\left(r^{h-1}[i] + 1\right) = \arg\min_{i \in N}\left(r^{h-1}[i] + 1\right) = \arg\min_{i \in N} r^{h-1}[i]$ for all $h = 1, ..., m$.

"$\Longrightarrow$" Suppose p be EF balanced, but it is not a gRR. Let be the earliest round $p^{[k]} = (p_{nk+1}, ...p_{(n+1)k})$ be the earliest round in which some agent, say i, does not appear Since rounds contain n agents each, there is an agent j who appears there at lest twice. Let h be the position where agent j appears for the second time in $p^{[k]}$. But then $r^{h-1}[i] = k < r^{h-1}[j] = k + 1$, a contradiction.

[7] We could further narrow it down to a singleton using some mild consistency/anonymity type requirement, as is traditionally done in the literature on apportionment methods.

"$\Longleftarrow$" Suppose p is a gRR. Let $h = nk + t$, $0 \le t < n$, be a position within k-th round $p^{[k]} = (p_{nk+1}, ..., p_{(n+1)k}) = (i_1, ..., i_n)$. Then, for any agent i_l we have $r^{h-1}[i_l] = k$ if $l \le t$, and $r^{h-1}[i_l] = k + 1$ if $l > t$. Hence, $p_h = i_t \in \arg\min_{i \in N} r^{h-1}[i]$. ■

To see that fair queues guarantee fair allocations, we will need the following equivalence result. As before, indifferences make the argument much more complicated, so we give the proof for strict preferences first.

We use the following notation. Let $S \subset A$, and $\overline{A} = (a_1, ..., a_n)_{\succeq}$, the vector of objects in A, ordered according to $\succeq$, as $a_1 \succeq ... \succeq a_n$. We define $R^h(S) \stackrel{def}{=} S \cap \{a_1, ..., a_h\}$, the elements of S which are among top h objects from A according to $\succeq$.

Lemma 5 *Let A be a set of m objects, $B, C \subset A$, and $\succeq$ be an ordering of A. Let $B = \{b_1, ..., b_q\}$, $C = \{c_1, ..., c_k\}$, ordered so that $b_1 \succeq ... \succeq b_q$, $c_1 \succeq ... \succeq c_k$. Order elements of A in decreasing order of $\succeq$ as $\overline{x} = (x_1, ..., x_m)$, with $x_1 \succeq ... \succeq x_m$, and so that objects from B appear in $\overline{x}$ as early as possible while objects from T appear in $\overline{x}$ as late as possible. Specifically, if $b \sim d \sim c$, $b \in B \backslash C$, $d \in B \cap C$, $c \in C \backslash D$, then a is before d and c, while d is before c.*

The following two statements are equivalent:

(1) $\frac{1}{q}|R^h(B)| \ge \frac{1}{k}|R^h(C)|$ for all $h = 1, ..., m$, where $R^h(S) = S \cap \{x_1, ..., x_h\}$.

(2) $\mathrm{Exp}_v\, B = \frac{1}{q} \sum_{b \in B} v(b) \ge \mathrm{Exp}_v\, C = \frac{1}{k} \sum_{c \in C} v(c)$ for any $v : A \to \mathbb{R}$ which represents $\succeq$.

Proof. *(1) $\Rightarrow$ (2)* It is easy to see that any non-increasing function $v : A \to \mathbb{R}$ can be represented as $v(x) = \sum_{h=1}^{n-1} \mu_h \chi_h(x) + \gamma$, where $\chi_h : A \to \mathbb{R}$ are characteristic functions of $\{x_1, ..., x_h\}$ (i.e., $\chi_h(x_i) = 1$ if $i \le h$ and $\chi_h(x_i) = 0$ otherwise), $\mu_h = v(x_h) - v(x_{h+1}) \ge 0$, and $\gamma = v(x_n)$ is a constant.

Due to additivity of expectation operator, it is enough to show that (2) implies (1) for $v = \chi_h$. But this is immediate:
$\mathrm{Exp}_{\chi_h} B = \frac{1}{q} \sum_{b \in B} \chi_h(b) = \frac{1}{q}|R^h(B)| \ge \frac{1}{k}|R^h(C)| = \frac{1}{k} \sum_{a \in A} \chi_h(a) = \mathrm{Exp}_{\chi_h} C$.

(2) $\Rightarrow$ (1) (I) (For a strict ordering $\succ$)

Suppose to the contrary, that for some h we have $\frac{1}{k}|R^h(C)| = \frac{1}{q}|R^h(B)| + \sigma$ for some $\sigma > 0$. Let v representing $\succ$ be such that $v(x) \in [1-\varepsilon, 1]$ for $x \in \{x_1, ..., x_h\}$ and $v(x) \in [0, \varepsilon]$ otherwise, where $\varepsilon > 0$ is very small. We have $\frac{1}{q}\sum_{b\in B} v(b) \leq \frac{1}{q}|R^h(B)| + \frac{1}{q}(q - |R^h(B)|)\varepsilon \leq \frac{1}{q}|R^h(B)| + \varepsilon$, while $\frac{1}{k}\sum_{c\in C} v(c) \geq \frac{1}{k}|R^h(C)|(1-\varepsilon) \geq \frac{1}{k}|R^h(C)| - \varepsilon$. Choosing $\varepsilon < \frac{1}{2}\sigma$, we obtain a desired contradiction to (2).

(2) ⇒ (1) (II) (For an arbitrary ordering $\succeq$)

Extending notation, we will use $|R^0(S)| \overset{def}{=} 0$, $|R^M(S)| \overset{def}{=} |R^m(S)| = |S|$ for $M > m$.

Fix h. Let $H = \{x_{r_1}, ..., x_{r_2}\} = \{a \in A : a \sim x_h\}$. Thus, $x_{r_1-1} \succ x_{r_1} \sim ... \sim x_{r_2} \succ x_{r_2+1}$, (it can be that $r_1 = 1$ and/or $r_2 = m$).

Using the same reasoning as in (I), we can show that $\frac{1}{q}|R^g(B)| \geq \frac{1}{k}|R^g(C)|$ for $g = r_1 - 1$ and $g = r_2$.

Let $H_B = H \cap (B\backslash C)$, $H_C = H \cap (C\backslash B)$, $H_{BC} = H \cap (B \cap C)$. In the ordering $\overline{x}$, the objects from H_B come first, followed by the objects from H_{BC}, and the last ones are the objects from H_C. Let x_l be the last in $\overline{x}$ object from H_B ($l = r_1 - 1$ if $H_B = \emptyset$), x_{f_1} and x_{l_1} be the first and last in $\overline{x}$ objects from H_{BC} (if $H_{BC} = \emptyset$ then let $f_1 = f - 1$ and $l_1 = l + 1$), and x_l be the first in $\overline{x}$ object from H_C ($f = r_2 + 1$ if $H_C = \emptyset$). Here $l < f$; if $H_{BC} \neq \emptyset$, we have $l < f_1 < l_1 < f$. Now:

$\frac{1}{q}|R^g(B)| \geq \frac{1}{q}|R^{r_1-1}(B)| \geq \frac{1}{k}|R^{r_1-1}(C)| = \frac{1}{k}|R^g(C)|$ for $g \in \left\{r_1 - 1, ..., f_1 - 1 \overset{def}{=} g_1\right\}$, since we have $R^g(B) \supset R^{r_1-1}(B)$, $R^{r_1-1}(C) = R^g(C)$.

$\frac{1}{q}|R^g(B)| = \frac{1}{q}|R^{r_2}(B)| \geq \frac{1}{k}|R^{r_2}(C)| \geq \frac{1}{k}|R^g(C)|$ for $g \in \left\{l_1 + 1 \overset{def}{=} g_2, ..., r_2\right\}$, since we have $R^g(B) = R^{r_1-1}(B)$, $R^{r_2}(C) \supset R^g(C)$.

If $H_{BC} = \emptyset$ then also $\frac{1}{q}|R^g(B)| \geq \frac{1}{k}|R^g(C)|$ for $f_1 - 1 \leq g \leq l_1 + 1$ (neither $R^g(B)$ nor $R^g(C)$ change when g moves from $f_1 - 1$ to $l_1 + 1$), and (1) is true.

Let $H_{BC} \neq \emptyset$. For any g, $g_1 \leq g \leq g_2$, we have $|R^g(B)| = |R^{g_1}(B)| + K_g$, $|R^g(C)| = |R^{g_1}(C)| + K_g$, where $K_g = |\{x_{g_1+1}, ..., x_g\} \cap H_{BC}|$.

Suppose $|R^{g_1}(B)| = 0$. Given that $\frac{1}{q}|R^{g_1}(B)| \geq \frac{1}{k}|R^{g_1}(C)|$, we have $|R^{g_1}(C)| = 0$, and so $|R^g(B)| = |R^g(C)|$ for $g_1 \leq g \leq g_2$. Since $\frac{1}{q}|R^{g_2}(B)| \geq \frac{1}{k}|R^{g_2}(C)| = \frac{1}{k}|R^{g_2}(B)|$, we have $\frac{1}{q} \geq \frac{1}{k}$. Hence, $\frac{1}{q}|R^g(B)| \geq \frac{1}{k}|R^g(B)| = \frac{1}{k}|R^{g_2}(C)|$, and (1) is true.

Suppose $|R^{g_1}(B)| > 0$.

We will use the following observation: Suppose Let $y \geq 0$, $z > 0$.

If $y \leq z$ then $y\delta \leq z\delta$, so $y(z+\delta) \leq z(y+\delta)$ and $\frac{y}{z} \leq \frac{y+\delta}{z+\delta}$ for any $\delta > 0$.

If $y \geq z$ then $y\delta \geq z\delta$, so $y(z+\delta) \geq z(y+v)$ and $\frac{y}{z} \geq \frac{y+\delta}{z+\delta}$ for any $\delta > 0$.

Thus, for any given y and z, $\frac{y+\delta}{z+\delta}$ is monotone in δ for $\delta > 0$. Hence, for any $\gamma \in (0, \Delta)$,
$\min\left\{\frac{y}{z}, \frac{y+\Delta}{z+\Delta}\right\} \leq \frac{y+\gamma}{z+\gamma} \leq \max\left\{\frac{y}{z}, \frac{y+\Delta}{z+\Delta}\right\}$.

Rewrite the above inequalities as $\frac{|R^{g_1}(C)|}{|R^{g_1}(B)|} \leq \frac{k}{q}$, $\frac{|R^{g_2}(C)|}{|R^{g_2}(B)|} = \frac{|R^{g_1}(C)|+K_{g_2}}{|R^{g^1}(B)|+K_{g_2}} \leq \frac{k}{q}$, for $g_1 = f_1 - 1 < l_1 + 1 = g_2$. By the observation above,
$\frac{|R^{g}(C)|}{|R^{g}(B)|} = \frac{|R^{g_1}(C)|+K_{g}}{|R^{g^1}(B)|+K_{g}} \leq \max\left\{\frac{|R^{g_1}(C)|}{|R^{g_1}(B)|}, \frac{|R^{g_2}(C)|}{|R^{g_2}(B)|}\right\} = \frac{k}{q}$, for any g, $g_1 \leq g \leq g_2$, so (1) is true. ■

We are now ready to demonstrate fairness of balanced queues.

Theorem 4 *(1) If a queue p is PROP balanced for agent i, then agent i satisfies oPROPu1 in (any allocation from) $\pi(p)$.*

(2) If a queue p is PROP' balanced for agent i, then agent i satisfies oPROP'u1 in (any allocation from) $\pi(p)$.

(3) If a queue p is EF balanced for agent i against agent j, then agent i is oEFu1 w.r.t. agent j in (any allocation from) $\pi(p)$.

(4) A queue p is PROP <PROP', EF w.r.t. agent j> balanced for agent i, iff it is oPROPu1 <oPROP', oEFu1 w.r.t. agent j> for any preferences $\succeq_i$.

Proof. To check (1) and (2), fix a preference profile $\succeq$ over A and a queue p. Recall that $r^h[i] = r^h(p)[i]$ is the number of sub-agents of i in the truncated queue p.

Note that, for any agent i and any $h = 1, ..., m$, no matter other agents' preferences, a sub-agent of i, whose turn in the queue p is no later then h, will always pick one of i'th top h objects.

Let $Z_i = \{z_1, ..., z_{q_i}\} = (z_1, ..., z_{q_i})_i$, with $z_1 \succeq_i ... \succeq_i z_{q_i}$, be the allocation for agent i in a $Z \in \pi(p)$; here her l-th sub-agent gets z_l. Let her post-upgrade share be $Z_i^u \in \arg\max\left\{u_i(Z_i), u_i(Z_i^{ba}) : a \in Z_i, b \notin Z_i\right\}$.

Order objects in A according to i's preferences, as $\overline{A} = (a_1, ..., a_m)_i$, with $a_1 \succeq_i ... \succeq_i a_m$, so that objects from Z_i appear in this ordering as early as possible and in the same order as $(z_1, ..., z_{q_i})_i$. As before, $R^h(S) = S \cap \{a_1, ..., a_h\}$. For any $h = 1, ..., m$, no matter other agents' preferences, $z_{r^h[i]} \succeq_i a_h$ (according to p, $z_{r^h[i]}$ is chosen by i when some of $\{a_1, ..., a_h\}$ are still available). Thus, $R^h(Z_i) \supset \{z_1, ..., z_{r^h(p)[i]}\}$ and $|R^h(Z_i)| \geq r^h[i]$. Further:

If $Z_i^u = Z_i$ (upgrades are not useful), then $Z_i^u = Z_i = \{a_1, ..., a_{q_i}\}$ – top q_i objects in $\overline{A}$. In this case, $|R^h(Z_i^u)| = |R(Z_i^u)| = \min\{h, q_i\} \geq \frac{q_i}{m}h$ (since both $h \geq \frac{q_i}{m}h$ and $q_i \geq \frac{q_i}{m}h$).

If $Z_i^u \neq Z_i$, we may assume $Z_i^u = \{z_1, ..., z_{q_i-1}\} \cup \{b\}$, where b is the earliest object in $\overline{A} = (a_1, ..., a_m)_i$ which does not belong to Z_i. Order $Z_i^u = (b_1, ..., b_{q_i})_i$ in the same way as how those objects appear in $\overline{A}$. Let $b = b_k$, $z_{q_i} = a_l$. Note that $(b_1, ..., b_k)_i = (a_1, ..., a_k)_i$, – top k objects in $\overline{A}$, so $b = b_k = a_k$.

PART (1). Let p be PROP balanced for i, so $r^h[i] + 1 \geq \frac{q_i}{m}h$ for all $h = 1, ..., m$.

Agent i is oPROPu1 iff $\frac{1}{q_i}\sum\limits_{z \in Z_i^u} v_i(z) = u_i(Z_i^u) \geq u_i(A) = \frac{1}{m}\sum\limits_{s \in A} v_i(s)$ for all v_i compatible with $\succeq_i$. By Lemma 5, it is equivalent to $\frac{1}{q_i}|R^h(Z_i^u)| \geq \frac{1}{m}|R^h(A)| = \frac{h}{m}$, or to $|R^h(Z_i^u)| \geq \frac{q_i}{m}h$, for all $h = 1, ..., m$.

So, if $Z_i^u = Z_i = (a_1, ..., a_{q_i})_i$, then oPROPu1 is true.

Suppose $Z_i^u = \{z_1, ..., z_{q_i-1}\} \cup \{b\}$.

Case 1 When $1 \leq h < k$, we have $R^h(Z_i^u) = Z_i^u \cap \{a_1, ..., a_h\} = \{a_1, ..., a_h\}$, so $|R^h(Z_i^u)| = h \geq \frac{q_i}{m}h$.

Case 2 When $k \leq h < l$, we have $R^h(Z_i^u) = R^h(Z_i) \cup \{b\}$, so $|R^h(Z_i^u)| = |R^h(Z_i)| + 1 \geq r^h[i] + 1 \geq \frac{q_i}{m}h$.

Case 3 When $l \leq h \leq m$, we have $R^h(Z_i^u) = Z_i^u$, so $|R^h(Z_i^u)| = q_i \geq \frac{q_i}{m}h$.

PART (2). Let p be PROP' balanced for i, so $r^h(p)[i] + 1 \geq \frac{q_i}{m}(h+1)$ for all $h = 1, ..., m$.

Agent i is oPROP'u1 iff $\frac{1}{q_i}\sum\limits_{z \in Z_i^u} v_i(z) = u_i(Z_i^u) \geq u_i(A \backslash Z_i) = \frac{1}{m-q_i}\sum\limits_{s \in A \backslash Z_i} v_i(s)$ for all v_i compatible with $\succeq_i$. By Lemma 5, it is equivalent to $\frac{1}{q_i}|R^h(Z_i^u)| \geq \frac{1}{m-q_i}|R^h(A \backslash Z_i)| = \frac{1}{m-q_i}(h - |R^h(Z_i)|)$, for all $h = 1, ..., m$. When $|R^h(Z_i^u)| = |R^h(Z_i)|$, this rewrites as $|R^h(Z_i^u)| \geq \frac{q_i}{m}h$.

If $Z_i^u = Z_i$, we have oPROP'u1.

Suppose $Z_i^u = \{z_1, ..., z_{q_i-1}\} \cup \{b\}$.

Case 1 When $1 \leq h < k$, we have $R^h(Z_i^u) = R^h(Z_i) = \{a_1, ..., a_h\}$, so $|R^h(Z_i^u)| = h \geq \frac{q_i}{m}h$, and we have oPROP'u1.

Case 2 When $k \leq h < l$, we have $R^h(Z_i^u) = R^h(Z_i) \cup \{b\}$, so $|R^h(Z_i^u)| = |R^h(Z_i)| + 1$. Thus, oPROP'u1 is equivalent to $\frac{1}{q_i}(|R^h(Z_i)| + 1) \geq \frac{1}{m-q_i}(h - |R^h(Z_i)|)$, or $(|R^h(Z_i)| + 1) \geq \frac{q_i}{m}(h+1)$. But this follows from PROP' balancedness, since $|R^h(Z_i)| \geq r^h(p)[i]$.

Case 3 When $l \leq h \leq m$, we have $R^h(Z_i^u) = Z_i^u$, so $|R^h(Z_i^u)| = |R^h(Z_i)| = q_i \geq \frac{q_i}{m}h$, and we have oPROP'u1.

PART (3). By Lemma 4 (1), we can assume $n = 2$. But, for $n = 2$, EF balancedness is the same as PROP' balancedness (Lemma 4 (2)), while EFu1 is the same as PROP'u1 (Lemma 2 (1)). So PART (3) follows from PART (2).

PART (4). Fix a queue p. Let all agents have the same binary preferences $\succeq^h$ with $a_1 \sim^h ... \sim^h a_h \succ^h a_{h+1} \sim^h ... \sim^h a_m$. Say, $v(a_k) = 1$ for $1 \leq k \leq h$ ("good" objects), $v(a_k) = 0$ for $k > h$ ("bad" objects). In any allocation resulting from p, first h sub-agents will get "good" objects, and others will get "bad" objects. Thus, for any agent i her utility from such allocation is $u_i = \frac{r^h(p)[i]}{q_i}$.

By contradiction, assume p does not satisfy one of our properties, so for some h the corresponding inequality is not satisfied. Pick corresponding preference profile $\succeq^h$, and let $Z = (Z_1, ..., Z_n) \in \pi(p)$ be an allocation resulting from p under those $\succeq^h$. Let Z_1^u be a post upgrade bundle for i, where she can at most substitute one "bad" object by one "good" one.

– Suppose p is not PROP balanced for an agent i, so $r^h(p)[i] + 1 < \frac{q_i}{m}h$. Then we have $u_i(Z_i^u) \leq \frac{r^h(p)[i]+1}{q_i} < \frac{1}{q_i}\frac{q_i}{m}h = \frac{1}{m}h = u_i(A)$, and no oPROPu1.

– Suppose p is not PROP' balanced for an agent i, so $\frac{1}{q_i}\left(r^h(p)[i] + 1\right) < \frac{1}{m-q_j}(h - r^h(p)[i])$. Then we have $u_i(Z_i^u) \leq \frac{r^h(p)[i]+1}{q_i} < \frac{1}{m-q_j}(h - r^h(p)[i]) = u_i(A \backslash Z_i)$, and no oPROP'u1.

– Suppose p is not EF balanced for an agent i w.r.t. an agent j, so $\frac{1}{q_i}\left(r^h(p)[i] + 1\right) < \frac{1}{q_j}r^h(p)[j]$. Then we have $u_i(Z_i^u) \leq \frac{r^h(p)[i]+1}{q_i} < \frac{1}{q_j}r^h(p)[j] = u_i(Z_j)$, and no oEFu1. ■

Finally, as a corollary of the theorems above, we can formulate our existence result: OE and ordinal fairness are compatible.

Theorem 5 *OE and oEFu1 (and hence oPROP'u1 or oPROPu1) allocations always exist. In particular, all Z in $\pi(p)$ for an EF balanced p satisfy those properties for any ordinal profile $\succeq$.*

Let us now discuss ordinal allocation rules $F : \succeq \longmapsto Z = (Z_1, ..., Z_n)$. We assume that a rule can be multi-valued, but it is sinlge-valued in utilities. Given an ordinal preference profile $\succeq$, it produces a set of allocations, such that any agent is ordinally (i.e., in the strong sense) indifferent between all allocations in this set.

As always, a rule is OE/PROP/EF, etc., iff any its outcome satisfies such property.

Given a preference profile $\succeq$, all OE allocations are generated by some queues. Hence, any OE allocation rule F can be described by specifying which queue is chosen for each preference profile. Let $f : \succeq \longmapsto p$. Then $F = F_f : \succeq \longmapsto \pi(f(\succeq))$.

Of course, a queue does not have to be fair to generate a fair allocation. For example, suppose agents have completely different preferences. Say, each agent i has a set S_i of exactly q_i objects as her top indifference class in her $\succeq_i$, and those sets S_i form a partition of A (i.e., they are disjoint). In this case, any queue would result in fully fair allocation $(S_1, ..., S_n)$, where each agent receives her best basket.

However, suppose that we want to use the same queue for all preferences profiles. This might be the case when we want to agree on a fair and efficient method of allocation before we know agents' preferences, or even before we know who are the agents. We could be, for example, writing an institutional policy, which then will be applied in a variety of situations. This defines the set of rules $\{F_p : p = (p_1, ..., p_n)\}$ to choose from.

Theorem 4 (3) tells us that, if we want to guarantee oEFu1 for all profiles, we have to choose an EF balanced queue. And such a queue is essentially unique (recall that it is unique up to $(m-n)$-th position when all fractions $\frac{r}{q_i}$, with $i \in N$, $1 \leq r \leq q_i$ are different). Thus:

Corollary 4.1

– *Each OE F is defined by $f : \succeq \longmapsto p = p(\succeq)$, with $F(\succeq) = \pi(p(\succeq))$.*

– There exist OE and oEFu1 (and hence oPROPu1) allocation rules $F : \succeq \longmapsto Z = Z(\succeq)$.

– Among rules F_p*, defined by* f *being a constant,* F_p *is oEFu1 iff* p *is EF balanced queue.*

4 Further Discussion

We believe that ordinal setting is an appropriate approach to the discrete fair division models. Ordinal design prevails in the real life, when full cardinal information is too reach. The advantages of simple input are twofold. Agents are not required to transmit or even to formulate complete cardinal preferences, but only need to figure out their ordering of options. Principal can propose a simple rule which is easily understood, and thus generates more trust among the agents. He also faces less computational problems.

In our setting, when the sizes of agents' bundles are fixed, this approach allowed us to describe efficient and fair rules. While similar ideas to achieve fairness by using queues were recently also considered in cardinal setting, efficiency did not was not discussed.

In the cardinal setting, under our fixed quotas assumption, the existence of fully efficient (PO) and fair (PROPu1 or EFu1) allocations is a difficult open problem. In particular, contrary to the traditional model, we can have fully efficient allocations where some pairs of agents envy each other.

Example 1 $N = \{1, 2\}$, $A = \{d, b, s\}$; $q_1 = 2$, $q_2 = 1$.
Let $u_1(d) = 10$, $u_1(b) = 9$, $u_1(d) = 0$; $u_2(18) = 18$, $u_2(b) = 1$, $u_2(d) = 0$.
Allocation $S_1 = \{d, s\}$, $S_2 = \{b\}$ is PO, but agents envy each other.

Combining PO and oPROPu1/oEFu1 is too strong. One cannot guarantee the existence of an allocation which would satisfy both.

Example 2 $N = N = \{1, 2\}$, $q_1 = q_2 = 4$, $A = \{a_1, ..., a_8\}$. Both agents have the same strict preferences $a_1 \succ ... \succ a_8$. Any PO Z allocation is also OE so it results from some queue p, i.e., $Z = \pi(p)$. If an allocation is also oEFu1, then this queue p

has to be a gRR. Thus, p consists of four rounds, and in each round agent 1 picks one object and agent 2 picks one object. Hence, each agent gets one object from each of four sets $\{a_1, a_2\}$, $\{a_3, a_4\}$,$\{a_5, a_6\}$,$\{a_7, a_8\}$. But this allocation is no PO for all valuations compatible with $\succ$.

Indeed, let agent 1 believe that only a_1, a_2 are valuable objects, while agent 2 think that $a_1, ..., a_6$ are valuable. Corresponding valuations (compatible with $\succ$) are as follows. Pick small $\varepsilon > 0$. Let $v_1(a_1), v_1(a_2) \in [1-\varepsilon, 1]$ and $v_1(a_3), ..., v_1(a_8) \in [0, \varepsilon]$, while $v_1(a_1), ..., v_1(a_6) \in [1-\varepsilon, 1]$ and $v_1(a_7), v_1(a_8) \in [0, \varepsilon]$.

We have $u_1(Z_1) \leq 1 + 3\varepsilon$, $u_2(Z_2) \geq 3 + \varepsilon$. For $\varepsilon < \frac{1}{5}$, both agents prefer allocation $Y_1 = \{a_1, a_2, a_7, a_8\}$, $Y_1 = \{a_3, a_4, a_5, a_6\}$, where they get $u_1(Y_1) \geq 2 - 2\varepsilon$, $u_2(Z_2) \geq 4(1-\varepsilon)$.

For our model, we propose new definitions of approximate fairness, stronger then standard PROP1/EF1, and more uniform – they apply equally well to cases of "goods", "chores", or mixed objects. One might ask whether at least some positive results from preceding cardinal setting literature (with no size constraints, or under different feasibility restrictions) still go through under those definitions. In the unconstrained setting though, up to one upgrade or up to one flip notions seem too strong. A simple example is when all agents are (almost) indifferent between all objects, and the number of objects is not a multiple of a number of agents, say $m = kn + r$, $0 < r < n$. The most fair way is to give some agents k objects and some others $k+1$ objects. An agent with k objects would envy anyone with $k+1$ objects, no matter how many "upgrades" or "flips" she is allowed.

Further, in unrestricted setting, "disregarding an object b outside a share S" and "adding to S an object $b \in S$" mean the same thing in terms of envy. However, under different quotas, it results in different properties. We could thus consider comparisons "up to one downgrade". For, example, define EFd1:

In an allocation Z agent i does not envy agent j "up to one downgrade", if either $u_i(Z_i) \geq u_i(Z_j)$ or there exists a "downgrading" Z_j^{ab} of Z_j (for some $a \in Z_i$, $b \in Z_j$) such that $u_i(Z_i) \geq u_i(Z_j^{ab})$.

EFd1 is neither stronger nor weaker than our EFu1; the relationship depends on which of q_i and q_2 is larger.

Example 3 Let $n = 2$, $m = 5$, $q_1 = 2$, $q_2 = 3$, $A = \{a, b, c, d, e\}$, and $v_1 = v_2 = v$ with $v(a) = v(b) = v(c) = 1$, $v(d) = 1.1$, $v(e) = 0.9$.

In the allocation $Z_1 = \{a, e\}$, $Z_2 = \{b, c, d\}$, agent 1 does not envy agent 2 after the upgrade to $Z_1^U = Z_1^{de} = \{a, d\}$. However, agent 1 would envy agent 2 even after the worst downgrade of Z_2 to $Z_2^D = Z_2^{ed} = \{b, c, e\}$.

In the allocation $Y_1 = \{a, d\}$, $Y_2 = \{b, c, e\}$, agent 2 does not envy agent 1 after after the downgrade of Z_1 to $Z_1^D = Z_1^{ed} = \{a, e\}$. However, agent 2 would envy agent 1 even after the best upgrade of Z_2 to $Z_2^U = Z_2^{de} = \{b, c, d\}$.

Up to one downgrade approximations seem less natural than our notions, but one might still wish to investigate their implications for various settings.

In our model with quotas, it gives us very similar results. We can define:

A queue p "EF^D balanced" is for i w.r.t. j iff $\frac{1}{q_i} r^h(p)[i] \geq \frac{1}{q_j}(r^h(p)[j] - 1)$.

The same reasoning as before gives us that:

– EF^D balanced queues are exactly those where each next $p^h \in \arg\min_{i \in N} \frac{1}{q_i} r^{h-1}(p)[i]$.

– If a queue is EF^D balanced for i w.r.t. j, then i oEFd1 w.r.t. j.

– If a queue guarantees oEFd1 for all preferences, then it is EF^D balanced.

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