An Axiomatic Characterization of Weighted Congestion Games

Vasily Gusev

WP HSE 1/EC/2025

ECONOMICS

HSE UNIVERSITY

NATIONAL RESEARCH UNIVERSITY
HIGHER SCHOOL OF ECONOMICS

An Axiomatic Characterization of Weighted Congestion Games

Vasily Gusev[1]

[1]HSE University, Russian Federation

Abstract

In weighted congestion games, players' weights and resource functions are predefined. This way, they can be applied for modeling traffic intensity, exploring market competition, and analyzing other problems with a congestion effect. In some normal-form games however, players' weights and resource functions are not defined explicitly and players may be unaware of their existence. This article finds the necessary and sufficient conditions for representing a normal-form game as a weighted congestion game. Axioms are formulated that guarantee there exist positive weights of players and positive-definite resource functions. It is proved that a normal-form game satisfies the axioms of Positivity and the Independence of Irrelevant Choices (Konishi et al., 1997) if and only if it is a singleton weighted congestion game. This result indicates that the payoff functions of players in hedonic games are represented in the form of a weighted congestion game. It is demonstrated that a normal-form game satisfies the axioms of Non-Negativity, Transfer, Resource Marginal Contribution, and the Independence of Irrelevant Choices if and only if it is a full weighted congestion game with player-independent resource functions.

Keywords: weighted congestion games, representation theorem, axiomatization

1. Introduction

1.1 Motivation questions

A weighted congestion game (Milchtaich, 1996) is a model of a game-theoretic process in which players exert positive or negative externalities on each other when choosing the same resources. Each player has a weight and the payoff from the chosen resource depends on the sum of weights of the players who have chosen it. If the resource payoff functions are monotonically decreasing, then a negative externalities effect is generated. The payoff of a player in a weighted congestion game is written in the additive form with each summand corresponding to the chosen resource. There are some game classes that broaden the perception of weighted congestion games (Mavronicolas et al., 2007; Milchtaich, 2009). Weighted congestion games model the processes of traffic flow distribution or manufacturing under competition (see Examples 1 and 2 in (Rosenthal, 1973)). The paper (Milchtaich, 2009) demonstrated the application of weighted congestion games in job balancing, M/M/1 queuing, and habitat selection.

The axiomatization of weighted congestion games is motivated by the following:

1. Modeling game-theoretic processes. Suppose some game-theoretic process possesses the Independence property. This property implies that the players do not influence each other if they choose different strategies. Weighted congestion games are known to possess this property. Is the Independence property alone sufficient for the game-theoretic process to be modeled by a weighted congestion game? Is it necessary to check for some other properties of the game-theoretic process in order to model it by a weighted congestion game? Characterization theorems contain answers to such questions.

2. Choosing the congestion model. Suppose there is a road network and we are interested in the road occupancy rate. Congestion games, weighted congestion games, weighted congestion games with player-specific payoff functions, and other games model this situation. In congestion games, each player has a unit weight and there exists a pure Nash equilibrium. In weighted congestion games however, players' weights differ and a pure equilibrium does not necessarily exist. How does one decide which game to use as a model for the game-theoretic process? How can one compose the resource functions and select players' weights? Knowing the axioms of the game, it suffices to check that they are satisfied in an applied problem to identify the suitable model.

3. Transfer to a model with independent parameters. In weighted congestion games, each player is associated with a certain weight and each resource with a payoff function. Players' weights are mutually independent and the payoff from one resource does not depend on the payoff from another. The independence of players' weights and resource functions is key to answering many questions. We can instantly say that the player with greater weight will have the greatest impact on payoffs from their chosen resource. In normal-form games however, players' weights and resource functions may not be explicitly set or may not exist. Players' payoff functions may also contain dependent parameters. Can we represent a game with dependent parameters as a weighted congestion game with independent parameters? The characterization of weighted congestion games provides the answer to this question as well.

If for some game-theoretic process the players' weights and resource functions are predefined, then we can apply the results of weighted congestion games to explore our question of interest. We can analyze whether an equilibrium exists, estimate the price of anarchy, rank players by weight, etc. In practice however, players' weights and resource functions may not be predefined. In a game-theoretic process, we know only the players, their strategies, and the payoffs for each strategy profile. The researcher or players may be unaware that the weights and resource functions exist. This article determines the conditions under which a normal-form game has players' weights and resource functions such that this game can be represented as a weighted congestion game.

1.2 Results

We characterize symmetric weighted congestion games (WCGs) in the cases where each player's strategy is to choose a single resource or to choose any subset of the set of resources. The game will be called a singleton in the former case and a full game in the latter case. We also characterize some games representing particular cases of WCGs. The results are given in Table 1. It shows how the characterization of WCGs changes as the resource functions are simplified and the number of players and resources decreases.

A characterization of singleton WCGs is given in Theorem 1. It shows that a normal-form game, Γ, from the given set of games satisfies the axioms of Positivity (P) and Independence of Irrelevant Choices (IIC) iff Γ is a singleton WCG. The axiom (P) implies the positivity of the payoff functions of players in the game Γ. The axiom (IIC) states the mutual independence of the payoffs of the players who employ different strategies. The axiom (IIC) was introduced in (Konishi et al., 1997a) in investigating whether a strong equilibrium exists in some game classes. The axioms (P) and (IIC) guarantee there exist positive weights and positive-definite resource functions such that the payoff functions of players in the game Γ can be represented as payoff functions of a WCG.

A normal-form game Γ from the given set of games is a full WCG iff Γ satisfies one of the axioms:

- Non-Negativity (NN), Resource Additivity (RA) (Theorem 2);
- Non-Negativity (NN), Transfer (T), Independence of Irrelevant Choice (IIC) (Theorem 3).

Theorems on the characterization of full WCGs are based on interrelations between the axioms set forth in Lemma 4.

The axiom (NN) states that a player's payoff is zero if the player has not chosen any resource. Otherwise, the player's payoff is positive. The (RA) axiom reflects the main property of WCGs, i.e., the player's payoff is additive when the strategy profile is decomposed into independent profiles. The axiom (T) allows the calculation of the payoff of each player when their chosen resources are combined or intersect. The meaning of the axioms is described in more detail below.

The WCG with player-independent resource functions (WCGI) is a WCG where the resource functions do not depend on the players. A characterization of singleton WCGIs is given in Theorem 4. We demonstrate that a game Γ from the given game set satisfies the axioms (P), (S), and (IIC) iff Γ is a WCGI. The axiom (S) implies that if players choose the same strategies, they get equal payoffs.

A characterization of singleton WCGIs with an extended set of strategies is given in Theorem 5. For an extended set of strategies, a player is allowed not to choose any resource.

Table 1: The characterization of games in the cases where each player's strategy is to choose a single resource ($S = M$) and an arbitrary subset in the set of resources ($S = 2^M$).

The game	The set of strategies	Conditions for $\lvert M\rvert, \lvert N\rvert$	Axioms
WCG	M	$\lvert N\rvert = 1$ or $1 \leq \lvert M\rvert \leq 2$	(P)
		$\lvert N\rvert \geq 2, \lvert M\rvert \geq 3$	(P), (IIC)
	2^M	$\lvert N\rvert = 1$ or $\lvert M\rvert = 1$	(NN)
		$\lvert N\rvert \geq 2, \lvert M\rvert \geq 2$	(NN), (RA)
			(NN), (T), (IIC)
WCGI	M	$\lvert N\rvert = 1, \lvert M\rvert \geq 1$	(P)
		$\lvert N\rvert \geq 2, 1 \leq \lvert M\rvert \leq 2$	(P), (S)
		$\lvert N\rvert \geq 2, \lvert M\rvert \geq 3$	(P), (S), (IIC)
	2^M	$\lvert N\rvert = 1, \lvert M\rvert = 1$	(NN)
		$\lvert N\rvert = 1, \lvert M\rvert \geq 2$	(NN), (RA)
		$\lvert N\rvert \geq 2, \lvert M\rvert = 1$	(NN), (S)
		$\lvert N\rvert \geq 2, \lvert M\rvert \geq 2$	(NN), (RA), (S)
			(NN), (RA), (RMC)
			(NN), (T), (S), (IIC)
			(NN), (T), (RMC), (IIC)
WCGC	M	$\lvert N\rvert = 1$ or $\lvert M\rvert = 1$	(P)
		$\lvert N\rvert = 2, \lvert M\rvert = 2$	(P), (PM)
		$\lvert N\rvert \geq 3, \lvert M\rvert = 2$	(P), (PM), (PS)
		$\lvert N\rvert \geq 2, \lvert M\rvert \geq 3$	(P), (PM), (PS), (IIC)
	2^M	$\lvert N\rvert = 1, \lvert M\rvert = 1$	(NN)
		$\lvert N\rvert = 1, \lvert M\rvert \geq 2$	(NN), (RA)
		$\lvert N\rvert \geq 2, \lvert M\rvert = 1$	(NN), (PS)
		$\lvert N\rvert \geq 2, \lvert M\rvert \geq 2$	(NN), (RA), (PS), (PM)
			(NN), (T), (PS), (PM), (IIC)

We demonstrate that a game Γ from the given game set satisfies the axioms (NN), Resource Marginal Contribution (RMC) and (IIC) iff Γ is a WCGI. The axiom (RMC) implies that the chosen resources generate equal contributions for players.

A normal-form game Γ from the given set of games is a full WCGI iff Γ satisfies one of the above axiom sets:

- Non-Negativity (NN), Resource Additivity (RA), Symmetry (S) (Theorem 6);
- Non-Negativity (NN), Transfer (T), Symmetry (S), Independence of Irrelevant Choice (IIC) (Theorem 7);
- Non-Negativity (NN), Resource Additivity (RA), Resourse Marginal Contribution (RMC) (Theorem 8);
- Non-Negativity (NN), Transfer (T), Resourse Marginal Contribution (RMC), Independence of Irrelevant Choice (IIC) (Theorem 9).

A characterization of singleton and full weighted congestion games with player constants (WCGCs) is given in Theorems 10–12. Axioms are introduced to describe how players' payoffs are related to payoffs dependent of maximal congestion profiles. A strategy profile is called a maximal congestion profile if every player has chosen the same strategy.

2. Literature review

Subsection 2.1 explains how our study differs from the papers that explore the isomorphism of normal-form games. Subsection 2.2 shows how our study is related to other works.

2.1 Isomorphism theorem and characterization theorem

The definition of isomorphic games is introduced in (Monderer & Shapley, 1996). The game definition and its components is given in Section 3. The games $\Gamma_1 = (N, \{S_i\}_{i\in N}, \{u_i\}_{i\in N})$ and $\Gamma_2 = (N, \{X_i\}_{i\in N}, \{h_i\}_{i\in N})$ are isomorphic if there exist bijections $g_i : S_i \to X_i \ \forall i \in N$, such that for every $i \in N$ and $\forall (s_1, s_2, ..., s_n) \in \prod_{j\in N} S_j$ we have

$$u_i(s_1, s_2, ..., s_n) = h_i(g_1(s_1), g_2(s_2), ..., g_n(s_n)).$$

It is proved in (Monderer & Shapley, 1996) that any potential game is isomorphic to congestion games. This result was strengthened in (Monderer, 2007) by proving the isomorphism of any strategic game and the q-congestion game for a given q. It was also demonstrated that a q-congestion game is isomorphic to a network game.

It is proved in (Bilò, 2007) that the class of normal-form games coincides with the class of congestion games with player-specific payoffs.

It is demonstrated in (Milchtaich, 2013) that any finite normal-form game is isomorphic to some weighted game on a graph. This result was used in (Milchtaich, 2021) to describe the topological properties of WCGs.

The results on the isomorphism of normal-form games and WCGs imply that the strategy sets in the original and isomorphic games may be different. In contrast to the papers mentioned above, we do not change the set of player strategies. We characterize the axioms that guarantee that a normal-form game can be represented as a WCG. This implies the existence of player weights and resource functions such that the player's payoff in a given finite game can be represented as a WCG. Each result is formulated as a characterization theorem (Thomson, 2001).

2.2 Relationship with other studies

The article (Milchtaich, 1996) introduces (unweighted) congestion games in which the player strategy sets are identical and the payoff of the player i is $u_i(\sigma_1, \sigma_2, ..., \sigma_n) = S_{i\sigma_i}(n_{\sigma_i})$, where n_{σ_i} is the number of players who have chosen the strategy σ_i from the profile $(\sigma_1, \sigma_2, ..., \sigma_n)$, $S_{i\sigma_i}$ is the utility function of the player i when choosing the strategy σ_i. The weighted congestion game was introduced in Section 8 of (Milchtaich, 1996) through generalization of the (unweighted) congestion game. The assumption is that the player i has a weight β_i and the utility function argument depends on the sum of weights of the players who have chosen the same strategy as the player i. The utility of the player i when employing the strategy σ_i is $S_{i\sigma_i}\left(\sum\limits_{j:\sigma_j=\sigma_i} w_j\right)$. As opposed to (Milchtaich, 1996), we are interested in the characterization of WCGs. We also consider the cases where the player's strategy is not a singleton set. Be warned that the notations here and in (Milchtaich, 1996) are different.

(Konishi et al., 1997a) introduced the axiom (IIC) and demonstrated that the axioms (IIC), Anonymity, and Partial Rivalry guarantee the existence of a strong Nash equilibrium in pure strategies in some games. In our study, the axiom (IIC) plays an important role in the characterization of WCGs.

There is a subtle connection between congestion games and the Shapley value, which permits the use of classical cooperative game theory methods for the characterization of WCGs. On the one hand, potential games are isomorphic to congestion games (Theorem 3.2. from (Monderer & Shapley, 1996)). On the other hand, players' payoff functions in a potential game can be represented as the Shapley value (Theorem 2 from (Ui, 2000)) of a cooperative game. Our characterization of WCGs uses, for example, the property of strategy profile decomposition, which is equivalent to the basis set expansion of a function in cooperative theory. A fairly wide range of axioms for the characterization of values for cooperative games with a coalition structure is presented in (Gómez-Rúa & Vidal-Puga, 2010).

Although there is a connection between congestion games and the Shapley value, the characterization of WCGs is notably different from the axioms of cooperative game values. For normal-form games we prove the existence of weights and resource functions, whereas in the cooperative game theory proof is in most cases concerned with the existence of weights only (Kalai & Samet, 1987). Furthermore, the characteristic functions of cooperative games are set in advance. In some cases, players' weights are predefined and involved in axiom (see, for example, axioms A4, $\hat{A}3$ from (Nowak & Radzik, 1995)). In this article, players' weights and resource functions are not used in axioms.

The papers (Hollard, 2000; Konishi et al., 1997a, 1997b) introduce axioms for normal-form games. The axioms are similar to the properties of weighted games that provide sufficient conditions for the existence of an equilibrium. The axioms in our work are also formulated for normal-form games, but there are substantial differences from these papers. We do not require the property of monotonicity of players' payoff functions for characterization of WCGs.

3. Key notations and definitions

A normal-form game with identical player strategy sets is a three-tuple $\Gamma = (N, S, \{u_i\}_{i\in N})$, where $N = \{1, 2, ..., n\}$ is the set of players, S is the finite set of strategies of each player from N, a mapping $u_i : S^n \to \mathbb{R}$ is the payoff function for the player $i \in N$.

A strategy profile $s \in S^n$ is a vector $s = (s_1, s_2, ..., s_n)$, where $s_i \in S$ is the strategy of the player $i \in N$. Seeking to isolate the strategy s_i of the player i in the profile $s \in S^n$, we write $s = (s_i, s_{-i})$, where $s_{-i} \in S^{n-1}$. Similarly, the strategies of the different players i and k are isolated by writing $s = (s_i, s_k, s_{-ik})$, where $s_i, s_k \in S, s_{-ik} \in S^{n-2}$.

The weighted congestion model is a tuple $(N, M, S, \{w_i\}_{i\in N}, \{c_{ij}\}_{ij\in N\times M})$, where N and S have the same meaning as in the game Γ; M is a finite set of resources; $w_i > 0$ is the weight of the player i; the resource function $c_{ij} : W \to \mathbb{R}^+$ is the payout function for the player i from the resource $j \in M$, where $W = \{\sum_{r\in K} w_r\}_{\substack{K\subseteq N\\ K\neq\emptyset}}$ is the set of all possible different sums of players' weights. For example, if $N = \{1, 2\}, w_1 = 5, w_2 = 6$, then $W = \{5, 6, 11\}$. If $w_1 = 5, w_2 = 5$, then $W = \{5, 10\}$.

A weighted congestion game (WCG) is a normal-form game $(N, S, \{u_i\}_{i\in N})$ in which $S \subseteq 2^N$ and the players' payoff functions have the form

$$u_i(s) = \sum_{j\in s_i} c_{ij}\left(\sum_{r\in K_j(s)} w_r\right),$$

where $K_j(s)$ is the set of players who have chosen the resource $j \in M$ in the profile $s \in S^n$.

A weighted congestion game with player constants (WCGC) is a WCG in which $c_{ij}(w) = \alpha_i \cdot c_j(w)\ \forall(i, j) \in N \times M\ \forall w \in W$, where $\alpha_i > 0$ is a constant for the player i and $c_j : W \to \mathbb{R}^+$. The payoff functions for players in a WCGC have the form

$$u_i(s) = \alpha_i \cdot \sum_{j\in s_i} c_j\left(\sum_{r\in K_j(s)} w_r\right).$$

A weighted congestion game with player-independent resource functions (WCGI) is a WCGC for which $\alpha_i = 1\ \forall i \in N$, that is, the players' payoff functions have the form

$$u_i(s) = \sum_{j\in s_i} c_j\left(\sum_{r\in K_j(s)} w_r\right).$$

Suppose the strategy set S somehow depends on M. We use $G(N, M, S)$ to denote a set consisting of all possible normal-form games $(N, S, \{u_i\}_{i\in N})$. The cases of interest are where $S = M$ or $S = 2^M$, called a *singleton* or a *full* game, respectively. The respective player strategy sets are called by analogy. In the former case, the player's strategy is to choose a single resource from M. In the latter case, the player is free to choose an arbitrary set of resources, and at that $\emptyset \in 2^M$. The strategy $s_i = \emptyset$ means that the player i has not selected any resource. Theoretically, it may not be beneficial for the players not to choose any resource, but this option is available to them in full games.

4. Characterization of weighted congestion games

4.1 Singleton WCGs

We begin this section by formulating the axioms (P) and (IIC). This subsection finds the necessary and sufficient conditions for the game $\Gamma \in G(N, M, S)$ to be represented as a singleton WCG.

Positivity (P). Let $\Gamma \in G(N, M, S)$. For any strategy profile $s \in S^n$ and for any player $i \in N$ we have

$$u_i(s) > 0.$$

The payoff of each player for each profile is positive.

Independence of Irrelevant Choices (IIC). Let $\Gamma \in G(N, M, S)$. For any two different players $i, k \in N$, any profile $s = (s_i, s_k, s_{-ik}) \in S^n$, any strategy $s'_k \in S$ such that $s_i \cap s_k = \emptyset, s_i \cap s'_k = \emptyset$ we have

$$u_i(s_i, s_k, s_{-ik}) = u_i(s_i, s'_k, s_{-ik}).$$

The axiom (IIC) was introduced in (Konishi et al., 1997a) while exploring the question of the existence of equilibrium in resource games. If the player k chooses the resources not chosen by the player i, this will not influence the playoff of the player i.

A characterization of singleton WCGs under the constraints $|N| \geq 2$ and $|M| \geq 3$ is given in Theorem 1. The rest of the cases are considered at the end of this subsection.

Theorem 1. *Let $|N| \geq 2, |M| \geq 3, S = M$. The game $\Gamma \in G(N, M, S)$ satisfies the axioms (P) and (IIC) iff there exists an array of positive player weights $\{w_i\}_{i \in N}$ and an array of resource functions $\{c_{ij}\}_{i \in N, j \in M}, c_{ij} : W \to \mathbb{R}^+ \ \forall (i, j) \in N \times M$ such that in the game Γ for $\forall i \in N \ \forall s \in S^n$ we have*

$$u_i(s) = c_{is_i}\left(\sum_{r \in K_{s_i}(s)} w_r\right).$$

According to Theorem 1, the game Γ can be represented as a singleton WCG iff Γ satisfies the axioms (P) and (IIC). This set of axioms guarantees the existence of positive weights of players and positive-definite resource functions. If only the resource function positivity is dropped, then for the game Γ to be represented as a WCG with positive weights of players it is necessary and sufficient to satisfy the axiom (IIC).

The proof of Theorem 1 provides an explicit representation of resource functions, and their uniqueness is demonstrated using the axiom (IIC). What is interesting about the proof of Theorem 1 is that the player weights are independent of the payoff functions for players of the game Γ. The only constraint on weights apart from their positivity is that all components of the vector $\left(\sum_{r \in K} w_r\right)_{\substack{K \subseteq N \\ K \neq \emptyset}}$ must be pairwise different. Knowing the characterization of WCGs, we can substantiate its application for modeling game-theoretic processes.

(Dreze & Greenberg, 1980) studies the "hedonic aspect" of coalition games, which implies that a player's payoff depends only on the coalition they have joined. To demonstrate the application of Theorem 1 to hedonic games, we can move from a coalition to astrategic form of games. This can be done simply by introducing n positions, with each player's strategy being to choose a position. The payoff of the player $i \in N$ depends only on the players who have chosen the same strategy as the player i. In this case, the hedonic aspect is equivalent to the axiom (IIC). If player utilities are positive, then, according to Theorem 1, we can represent the hedonic game as a WCG with positive weights of players and positive-definite resource functions.

Generalizing the parameters of congestion games, we remain in the class of hedonic games. However, it follows from Theorem 1 that when moving from unit to arbitrary weights of players, we get a complete class of hedonic games.

The literature on WCGs contains many results on the existence and properties of equilibria, which can be applied for studying the types of stability in hedonic games (Bogomolnaia & Jackson, 2002). For example, (Bilò et al., 2023; Gusev et al., 2024) studied a WCG with a weight matrix and resource functions $c_{ij}(w) = w_{ij} \cdot \frac{r_j}{w}$ $\forall w \in W$, where r_j is the size of the resource $j \in M$ and w_{ij} is the weight of the player i for the resource j. For some particular cases, the existence of a pure Nash equilibrium has been proved.

Note that representation theorems can be used to prove the existence of equilibrium, for example, see Proposition 1 from (Milchtaich, 2009).

If we have a hedonic game with a weight matrix $(w_{ij})_{i\in,j\in M}$, then by Theorem 1, we can represent such a game as a weighted congestion game in which the player weights do not depend on the resource.

If $|N| = 1$ or $1 \leq |M| \leq 2$, the axiom (P) is necessary and sufficient for representing the game $\Gamma \in G(N, M, S)$ as a singleton WCG with positive-definite resource functions. Let us consider these cases.

Let $N = \{1\}, |M| \geq 1$. Here, the payoff of player 1 can be represented as $u_1(s_1) = c_{1s_1}(w_1)$, where, for example, $w_1 = 1, c_{1j}(1) = u_1(j)$ $\forall j \in M$.

Let $|N| \geq 2, M = \{a\}$. In this case, there exists the only strategy profile $s = (a, a, ..., a)$ and we can represent the payoff of each player $i \in N$ as $u_i(s) = c_{ia}(w_1 + w_2 + ... + w_n)$, where, for example, $w_i = 1, c_{ia}(n) = u_i(a, a, ..., a)$ $\forall i \in N$.

Let $|N| \geq 2, |M| = 2$. We set the values of weights and resource functions in the same way as in the proof of Theorem 1. The axiom (IIC) is not necessary for the uniqueness of resource functions, since the set $\overline{S}(K, j)$ defined in the proof of Theorem 1, will consist of a single profile.

4.2 Resource Additivity axiom: a characterization of full WCGs

In full WCGs, each player can choose an arbitrary set of resources from M, with an option of not choosing any resource. We formulate the axioms (RA) and (NN). Theorem 2 gives a characterization of full WCGs.

Resource Additivity (RA). Let $\Gamma \in G(N, M, S)$. For any strategy profiles $s = (s_k, s_{-k}) \in S^n, s' = (s'_k, s'_{-k}) \in S^n, s'' = (s''_k, s''_{-k}) \in S^n$, such that $s_k = s'_k \cup s''_k \ \forall k \in N$ and $(\cup_{k=1}^n s'_k) \cap (\cup_{k=1}^n s''_k) = \emptyset$, the players' payoff functions in the game Γ satisfy the equality

$$u_i(s) = u_i(s') + u_i(s'') \qquad \forall i \in N.$$

As applied to the axiom (RA), the condition $(\cup_{k=1}^n s'_k) \cap (\cup_{k=1}^n s''_k) = \emptyset$ implies the independence of the profiles s' and s'', that is there is no player who chooses one and the same resource in the profiles s' and s''. When the profile is decomposed into independent profiles, the player's payoff possesses the additivity property.

For example, let $N = \{1, 2, 3\}, M = \{a, b, c, d, e\}, S = 2^M$. Let us consider a game $\Gamma \in G(N, M, S)$ for which (RA) is satisfied. Suppose $s = (\{a, b\}, \{b, c, d\}, \{d, e\}) \in S^n$. In that case, we can decompose the profile s, for example, into the profiles $s' = (\{a, b\}, \{b, c\}, \emptyset)$ and $s'' = (\emptyset, \{d\}, \{d, e\})$. Hence, players' payoff functions in the game Γ satisfy the equality

$$u_i(\{a, b\}, \{b, c, d\}, \{d, e\}) = u_i(\{a, b\}, \{b, c\}, \emptyset) + u_i(\emptyset, \{d\}, \{d, e\}) \ \forall i \in N.$$

Continuing the decomposition process, we can get the following equality

$$u_i(\{a, b\}, \{b, c, d\}, \{d, e\}) = u_i(\{a\}, \emptyset, \emptyset) + u_i(\{b\}, \{b\}, \emptyset)$$

$$+u_i(\emptyset, \{c\}, \emptyset) + u_i(\emptyset, \{d\}, \{d\}) + u_i(\emptyset, \emptyset, \{e\}) \ \forall i \in N.$$

Each summand depends on one resource and the set of players who have chosen it. If players are allowed to choose several resources, then the axiom (RA) is a fundamental property for calculating players' payoffs in a WCG.

Non-Negativity (NN). Let $\emptyset \in S$ and $\Gamma \in G(N, M, S)$. For any player $i \in N$ and for any profile $s = (s_i, s_{-i}) \in S^n$ we have

$$u_i(s_i, s_{-i}) > 0 \Leftrightarrow s_i \neq \emptyset \text{ and } u_i(s_i, s_{-i}) = 0 \Leftrightarrow s_i = \emptyset.$$

A player's payoff is zero iff they have not chosen any resource. A player's payoff is positive if they have chosen at least one resource.

Theorem 2 provides a characterization of full WCGs under the constraints $|N| \geq 2$ and $|M| \geq 2$. It is easy to show that for a normal-form game to be represented as a WCG with positive-definite resource functions at $|N| = 1$ or $|M| = 1$ it is necessary and sufficient to satisfy the axiom (NN). The proof of Theorem 2 builds on Lemma 1.

Theorem 2. *Let $|N| \geq 2, |M| \geq 2, S = 2^M$. The game $\Gamma \in G(N, M, S)$ satisfies the axioms (NN) and (RA) iff there exists an array of positive player weights $\{w_i\}_{i \in N}$ and an array of resource functions $\{c_{ij}\}_{i \in N, j \in M}, c_{ij} : W \to \mathbb{R}^+ \ \forall (i, j) \in N \times M$ such that in the game Γ for*

$\forall i \in N \; \forall s \in S^n$ *we have*

$$u_i(s) = \sum_{j \in s_i} c_{ij} \left(\sum_{r \in K_j(s)} w_r \right)$$

when $s_i \neq \emptyset$ *and* $u_i(s) = 0$ *when* $s_i = \emptyset$.

Lemma 1 shows how the payoff of players in the game $\Gamma \in G(N, M, S)$ can be represented if Γ satisfies the axiom (RA).

Lemma 1. *Let* $S = 2^M$ *and* $\Gamma \in G(N, M, S)$. *Then, the following two statements are true:*
1. The game Γ *satisfies the axiom (RA) iff for* $\forall i \in N \; \forall s \in S^n$ *we have*

$$u_i(s) = \sum_{j \in \underset{k \in N}{\cup} s_k} u_i(s_1 \cap \{j\}, s_2 \cap \{j\}, ..., s_n \cap \{j\}).$$

2. If the game Γ *satisfies the axioms (NN) and (RA), then for* $\forall i \in N \; \forall s \in S^n$ *we have*

$$u_i(s) = \sum_{j \in s_i} u_i(s_1 \cap \{j\}, s_2 \cap \{j\}, ..., s_n \cap \{j\})$$

when $s_i \neq \emptyset$ *and* $u_i(s) = 0$ *when* $s_i = \emptyset$.

We can represent a normal-form game with a full set of player strategies as a full WCG iff the game satisfies the axioms (NN) and (RA).

The characterization of singleton WCGs given in Theorem 1 depends on the axiom (IIC), while the characterization of full WCGs in Theorem 2 does not explicitly depend on the axiom (IIC). We can say that the axiom (RA) comprises the Additivity property as well as the (IIC) property. In the next section, we introduce the axiom Transfer (T) and give Lemma 4 to show the relationships between the axioms (RA), (NN), (IIC), and (T) for games with a full set of strategies.

4.3 Transfer axiom: characterization of full WCGs without (RA)

This section introduces the axiom Transfer (T) and provides an alternative characterization of full WCGs.

Transfer (T). Let $\Gamma \in G(N, M, S)$. Then, for any two players $i, k \in N$ and for any profiles $(s_k, s_{-k}), (s'_k, s_{-k}) \in S^n$ such that $s_k \cup s'_k, s_k \cap s'_k \in S$ we have

$$u_i(s_k, s_{-k}) + u_i(s'_k, s_{-k}) = u_i(s_k \cup s'_k, s_{-k}) + u_i(s_k \cap s'_k, s_{-k}).$$

We can transform the equality of the axiom (T) as follows,

$$u_i(s_k \cup s'_k, s_{-k}) = u_i(s_k, s_{-k}) + u_i(s'_k, s_{-k}) - u_i(s_k \cap s'_k, s_{-k}).$$

In this case, the player i can calculate their utility from partitioning the player k's resource by the set addition formula written for the functions. Note that in the axiom (T) the players i and k may coincide.

If the "=" sign in the axiom (T) equality is replaced with "$\geq$", then we get the componentwise concavity property (Shapley, 1971) of the function u_i or some analog of the Monotonicity and Marginality properties (Chun, 1989; van den Brink & Pinter, 2015; Young, 1985). Since the WCG satisfies the equalities of the axiom (T), it would be excessive to require the satisfaction of the inequalities. An analog of the axiom (T) is used in the cooperative game theory, for example, for the characterization of the Shapley-Shubik index (see axiom 2 from (Dubey et al., 2005) and axiom II from (Einy & Haimanko, 2011)).

We formulate Theorem 3 on characterization of full WCGs without the axiom (RA). The proof of Theorem 3 is based on Theorem 2 and Lemma 4. Lemma 4 rests upon Lemmas 1-3, with Lemmas 2 and 3 formulated after Theorem 3.

Theorem 3. *Let $|N| \geq 2, |M| \geq 2, S = 2^M$. The game $\Gamma \in G(N, M, S)$ satisfies the axioms (NN), (T) and (IIC) iff there exists an array of positive player weights $\{w_i\}_{i \in N}$ and an array of resource functions $\{c_{ij}\}_{i \in N, j \in M}, c_{ij} : W \to \mathbb{R}^+ \; \forall (i, j) \in N \times M$ such that in the game Γ for $\forall i \in N \; \forall s \in S^n$ we have*

$$u_i(s) = \sum_{j \in s_i} c_{ij} \left(\sum_{r \in K_j(s)} w_r \right)$$

when $s_i \neq \emptyset$ and $u_i(s) = 0$ when $s_i = \emptyset$.

Proof. It is not hard to check that a game with the indicated player payoff functions satisfies the above axioms.

Let Γ satisfy the above axioms. Then, the existence of player weights and resource functions, as well as the representation of the payoffs of players in the game Γ as a WCG follows from Theorem 2 and Lemma 4. As per the condition, $S = 2^M$ and Γ satisfies the axioms (NN), (T), and (IIC). Then, according to Lemma 4, Γ satisfies the axiom (RA). Since Γ satisfies the axioms (NN) and (RA), then Theorem 2 entails the assertion of the theorem being proved. $\square$

Next, we formulate Lemmas 2–4. Lemma 2 shows how the payoff of each player in the game Γ can be represented if Γ satisfies the axioms (NN) and (T).

Lemma 2. *Let $S = 2^M, \Gamma \in G(N, M, S)$ and Γ satisfies the axioms (NN) and (T). Then for $\forall i \in N \; \forall s \in S^n$ we have*

$$u_i(s) = \sum_{j \in s_i} u_i(\{j\}, s_{-i})$$

when $s_i \neq \emptyset$ and $u_i(s) = 0$ when $s_i = \emptyset$.

We know from the axiom (IIC) that if the strategies of the different players i and k do not involve identical resources, the players have no influence on each other's payoffs. In Lemma 3, we expand the property of (IIC). Suppose the player i chose the strategy $\{j\}$ and $j \in s_k$. If a game with a full set of player strategies satisfies the axioms (T) and (IIC), then the payoff of the player i will not change if the player k uses the strategy $\{j\}$. For instance, $u_i(\{j\}, \{a, b, j\}, s_{-ik}) = u_i(\{j\}, \{j\}, s_{-ik})$.

Lemma 3. *Let $S = 2^M, \Gamma \in G(N, M, S)$ and Γ satisfies the axioms (T) and (IIC). Then for any different players $i, k \in N$ and any profile $(\{j\}, s_k, s_{-ik}) \in S^n$, such that $j \in s_k$ we have*

$$u_i(\{j\}, s_k, s_{-ik}) = u_i(\{j\}, \{j\}, s_{-ik}).$$

Lemma 4 shows the relationship between the axioms (NN), (T), (IIC) and (RA) for games with a full set of player strategies.

Lemma 4. *Let $S = 2^M$. Then we have the following property of the axioms:*

$$(NN), (T), (IIC) \Rightarrow (RA).$$

Theorems 2 and 3 provide a characterization of full WCGs. If a game Γ satisfies the relevant axioms, then for each player there exist player-specific positive weights and for the resources there exist positive-definite resource functions such that the payoff of players in the game Γ can be represented as payoffs of players in a WCG.

5. Characterization of WCGIs

The previous section showed which conditions are necessary and sufficient for singleton and full games to be representable as WCGs. In this section, we are interested in the characterization of WCGIs in which the utility from the chosen resource depends only on the resource *per se* and the sum of weights of the players who have chosen it.

5.1 The axioms of Symmetry and Resource Marginal Contribution

Let us begin this subsection by formulating the axioms of Symmetry (s) and Resource Marginal Contribution (RMC) and demonstrating the relationship between them in some game classes.

Symmetry (S). Let $\Gamma \in G(N, M, S)$. For any two different players $i, k \in N$ and any profile $s = (s_i, s_k, s_{-ik}) \in S^n$ such that $s_i = s_k$ the payoff functions for the players i and k in the game Γ satisfy the equality

$$u_i(s) = u_k(s).$$

If players choose identical strategies, they get identical payoffs.

In (Myerson, 1977) the axiom of Marginal Contribution for cooperative games is introduced. We introduce an analog of this axiom for normal-form games.

Resource Marginal Contribution (RMC). Let $\Gamma \in G(N, M, S)$. For any two different players $i, k \in N$, and for any resource $j \in M$, for any profile $s = (s_i, s_k, s_{-ik}) \in S^n$ such that

$j \in s_i, j \in s_k, s_i \setminus \{j\} \in S, s_k \setminus \{j\} \in S$ we have

$$u_i(s_i, s_k, s_{-ik}) - u_i(s_i \setminus \{j\}, s_k, s_{-ik}) = u_k(s_i, s_k, s_{-ik}) - u_k(s_i, s_k \setminus \{j\}, s_{-ik}).$$

If the game Γ satisfies the axiom (RMC), then each resource $j \in M$ generates equal contributions for the players who have chosen it.

Lemma 5. *Let $\Gamma \in G(N, M, S)$. The following statements are true:*

1. Let $\emptyset \in S, \{j\} \in S$ and Γ satisfies the axioms (NN) and (RMC). Then for any profile $s \in S^n$, such that $s_i \in \{\emptyset, \{j\}\}, s_k = s_i$ we have

$$u_i(s) = u_k(s).$$

2. Let $S = M \cup \{\emptyset\}$. Then we have the following property of the axioms:

$$(NN), (RMC) \Rightarrow (S).$$

Lemma 5 is involved in the characterization of singleton and full WCGIs.

WCGs have players and resources. The axioms (S) and (RMC) describe the properties of players' payoffs and the properties of resources. The players who choose the same strategies get the same payoffs. This property is most of all associated with players. Yet, we can reformulate this property saying that a resource cannot discriminate between players' and provides the same utility for all the players who have chosen it. Then, the axiom (S) relates more to the property of the resource.

According to the axiom (RMC), a resource provides equal contributions to players. This reflects the properties of players' payoffs and resources.

5.2 Characterization theorems

This subsection characterizes singleton and full WCGIs. A characterization of singleton WCGIs under the constraints $|N| \geq 2$ and $|M| \geq 3$ is given in Theorem 4. The rest of the cases are considered in subsection 5.3.

Theorem 4. *Let $|N| \geq 2, |M| \geq 3, S = M$. The game $\Gamma \in G(N, M, S)$ satisfies the axioms (P), (S), and (IIC) iff there exists an array of positive player weights $\{w_i\}_{i \in N}$ and an array of resource functions $\{c_j\}_{j \in M}, c_j : W \to \mathbb{R}^+ \; \forall j \in M$, such that in the game Γ for $\forall i \in N \; \forall s \in S^n$ we have*

$$u_i(s) = c_{s_i}\left(\sum_{r \in K_{s_i}(s)} w_r\right).$$

Since WCGIs are WCGs, the characterization of singleton WCGIs contains the axioms that characterize singleton WCGs, i.e., (P) and (IIC). It is clear from Theorem 4 that it is necessary and sufficient to require that a WCG satisfies the axiom (S) to get a WCGI.

Theorem 5. *Let $|N| \geq 2, |M| \geq 3, S = M \cup \{\emptyset\}$. The game $\Gamma \in G(N, M, S)$ satisfies the axioms (P), (RMC), and (IIC) iff there exists an array of positive player weights $\{w_i\}_{i \in N}$ and an array of resource functions $\{c_j\}_{j \in M}, c_j : W \to \mathbb{R}^+ \ \forall j \in M$, such that in the game Γ for $\forall i \in N \ \forall s \in S^n$ we have*

$$u_i(s) = c_{s_i}\left(\sum_{r \in K_{s_i}(s)} w_r\right).$$

when $s_i \neq \emptyset$ and $u_i(s) = 0$ when $s_i = \emptyset$.

Proof. It is not hard to show that a game with the specified player payoff functions satisfies these axioms.

Let Γ satisfy the axioms (NN), (RMC), and (IIC). We will show that the player payoff functions in the game Γ can be represented in the said form.

It follows from Theorem 4 that under $|N| \geq 2, |M| \geq 3, S = M$, the game Γ is a singleton WSGI iff Γ satisfies the axioms (P), (S), and (IIC). The addition of an $\emptyset$ strategy has no effect on the player weights and resource functions from the proof of Theorem 4. Then, it follows from point 2 of Lemma 5 that the axiom (S) can be replaced with the axioms (NN) and (RMC). We get an axiom set (P), (NN), (RMC), and (IIC) which characterizes a WCGI with the strategy set $S = M \cup \{\emptyset\}$. It is also sufficient to have the axiom (NN) for the resource function positivity, so we can remove the axiom (P) from the list. □

Theorem 5 leads us to the following corollary. If resources in singleton WCGs generate equal contributions for players, and players are free not to choose any resource, then the WCG is a WCGI.

Theorems 6-9 provide a characterization of full WCGIs under the constraints $|N| \geq 2$ and $|M| \geq 2$. The rest of the cases are considered in Subsection 5.3. Theorems 6 and 7 give a characterization of full WCGIs using the axiom (S).

Theorem 6. *Let $|N| \geq 2, |M| \geq 2, S = 2^M$. The game $\Gamma \in G(N, M, S)$ satisfies the axioms (NN), (RA), and (S) iff there exists an array of positive player weights $\{w_i\}_{i \in N}$ and an array of resource functions $\{c_j\}_{j \in M}, c_j : W \to \mathbb{R}^+ \ \forall j \in M$, such that in the game Γ for $\forall i \in N \ \forall s \in S^n$ we have*

$$u_i(s) = \sum_{j \in s_i} c_j\left(\sum_{r \in K_j(s)} w_r\right)$$

when $s_i \neq \emptyset$ and $u_i(s) = 0$ when $s_i = \emptyset$.

Theorem 7. *Let $|N| \geq 2, |M| \geq 2, S = 2^M$. The game $\Gamma \in G(N, M, S)$ satisfies the axioms (NN), (T), (S), and (IIC) iff there exists an array of positive player weights $\{w_i\}_{i \in N}$ and an array of resource functions $\{c_j\}_{j \in M}, c_j : W \to \mathbb{R}^+ \ \forall j \in M$, such that in the game Γ for $\forall i \in N \ \forall s \in S^n$ we have*

$$u_i(s) = \sum_{j \in s_i} c_j\left(\sum_{r \in K_j(s)} w_r\right)$$

when $s_i \neq \emptyset$ and $u_i(s) = 0$ when $s_i = \emptyset$.

Proof. It is not hard to check that a game with the specified player payoff functions satisfies these axioms.

Let Γ satisfy the listed axioms. Then, the existence of player weight and resource functions, as well as the representability of player payoffs in the game Γ as a WCGI follows from Theorem 6 and Lemma 4. As per the condition, $S = 2^M$ and Γ satisfies the axioms (NN), (T), and (IIC). Then, according to Lemma 4, Γ satisfies (RA). Since Γ satisfies the axioms (NN),(RA), and (S), then Theorem 6 entails the assertion of the theorem being proved.

□

Theorems 8 and 9 provide a characterization of full WCGIs using the axiom (RMC).

Theorem 8. *Let $|N| \geq 2, |M| \geq 2, S = 2^M$. The game $\Gamma \in G(N, M, S)$ satisfies the axioms (NN), (RA), and (RMC) iff there exists an array of positive player weights $\{w_i\}_{i \in N}$ and an array of resource functions $\{c_j\}_{j \in M}, c_j : W \to \mathbb{R}^+ \ \forall j \in M$, such that in the game Γ for $\forall i \in N \ \forall s \in S^n$ we have*

$$u_i(s) = \sum_{j \in s_i} c_j \left(\sum_{r \in K_j(s)} w_r \right),$$

when $s_i \neq \emptyset$ and $u_i(s) = 0$ when $s_i = \emptyset$.

Proof. It is not hard to show that a game with the specified player payoff functions satisfies these axioms. Theorem 8 follows from Theorem 6 and point 2 of Lemma 5.

□

Theorem 9. *Let $|N| \geq 2, |M| \geq 2, S = 2^M$. The game $\Gamma \in G(N, M, S)$ satisfies the axioms (NN), (T), (RMC), and (IIC) iff there exists an array of positive player weights $\{w_i\}_{i \in N}$ and an array of resource functions $\{c_j\}_{j \in M}, c_j : W \to \mathbb{R}^+ \ \forall j \in M$, such that in the game Γ for $\forall i \in N \ \forall s \in S^n$ we have*

$$u_i(s) = \sum_{j \in s_i} c_j \left(\sum_{r \in K_j(s)} w_r \right),$$

when $s_i \neq \emptyset$ and $u_i(s) = 0$ when $s_i = \emptyset$.

Proof. The proof is similar to the proof of Theorem 3. Theorem 9 follows from Theorem 8 and Lemma 4. □

We can conclude from Theorems 4–9 that for symmetric resource functions to exist in singleton and full WCGs it is necessary and sufficient to satisfy the axiom (S) or (RMC).

5.3 Simple cases of WCGI characterization for small $|N|$ and $|M|$

Theorem 4 characterizes singleton WCGIs under the conditions $|N| \geq 2, |M| \geq 3$. Let us examine the cases for the $|N|$ and $|M|$ that are not covered by Theorem 4: 1. $|N| = 1$; 2. $|N| \geq 2, |M| = 1$; 3. $|N| \geq 2, |M| = 2$.

1. Let $N = \{1\}$. Then there exist numbers $w_1, \{c_j(w_1)\}_{j \in M}$ such that $u_1(s_1) = c_{s_1}(w_1)\ \forall s_1 \in S$. For example, $w_1 = 1, c_j(w_1) = u_1(j)\ \forall j \in M$. In this case, it is necessary and sufficient to satisfy only the axiom (P).

2. Let $|N| \geq 2, S = M = \{a\}$. Then the game Γ has only one strategy profile $s = (a, a, ..., a)$. It follows from the axiom (S) that $u_i(s) = u_k(s)\ \forall i, k \in N$. We can represent each player's payoff for the profile s as $u_i(s) = c_a(w_1 + w_2 + ... + w_n)$, where, for example, $w_i = 1\ \forall i \in N, c_a(w_1 + w_2 + ... + w_n) = c_a(n) = u_1(s)$. Since Γ satisfies the axiom (P), $c_a(n)$ takes a positive value.

3. Let $|N| \geq 2, |M| = 2$. We define players' weights and resource function values in the same manner as in the proof of Theorem 4. The difference is that for $|M| = 2$ the set $\overline{S}(K, j)$ consists of a single profile. Hence, the axiom (IIC) is not required for resource function uniqueness.

Theorem 6 characterizes full WCGIs under the conditions $|N| \geq 2, |M| \geq 2$. Let us consider the cases for the $|N|$ and $|M|$ not covered by Theorem 6: 1. $|N| = 1, |M| = 1$; 2. $|N| = 1, |M| \geq 2$; 3. $|N| \geq 2, |M| = 1$.

1. Let $N = \{1\}, M = \{a\}$, that is $|N| = |M| = 1$. Then $S = 2^M = \{\emptyset, \{a\}\}$. We can represent the payoff of player 1 as $u_1(\emptyset) = 0$ and $u_1(\{a\}) = c_a(w_1)$, where $w_1 = 1, c_a(1) = u_1(\{a\})$. In this case, it is necessary and sufficient to satisfy only the axiom (NN).

2. Let $N = \{1\}, |M| \geq 2$. Then for positive weights and positive-definite resource functions to exist it is necessary and sufficient to satisfy the axioms (NN) and (RA). We get that $u_1(\{j\}) = c_j(w_1)\ \forall j \in M$ and $u_1(s_1) = \sum_{j \in s_1} c_j(w_1) = \sum_{j \in s_1} u_1(\{j\})\ \forall s_1 \in 2^M, s_1 \neq \emptyset$. Then, for example, $w_1 = 1, c_j(1) = u_1(\{j\})\ \forall j \in M$.

3. Let $|N| \geq 2, M = \{a\}, S = 2^M = \{\emptyset, \{a\}\}$. We define player weights and the values of the resource function c_a in the same manner as in point 1 of Theorem 6. For the function c_a to be defined uniquely and take positive values it is enough to satisfy the axioms (NN) and (S). We do not use the axiom (RA) since $S = \{\emptyset, \{a\}\}$ does not contain any strategies that correspond to the choice of two or more resources.

6. Characterization of WCGCs

6.1 Proportion axioms with maximal congestion

This subsection introduces axioms that reflect the proportion properties of player payoffs in normal-form games. The proportionality principle often occurs in resource allocation problems (Besner, 2019; Zou et al., 2021). The axioms describe the relationship between player payoffs for standard profiles and maximal-congestion profiles. We call a profile $s \in S^n$ a *maximal-congestion profile* if $s_i = s_k\ \forall i, k \in N$, that is if all players have chosen the same strategy. In other words, the chosen resources are maximally congested.

Proportional symmetry with maximal congestion (PS). Let $\Gamma \in G(N, M, S)$. For any different players $i, k \in N$, for any strategy $j \in S$ such that $u_l(j, j, ..., j) \neq 0\ \forall l \in N$ and any strategy profile $s = (j, j, s_{-ik}) \in S^n$ we have

$$\frac{u_i(j, j, s_{-ik})}{u_i(j, j, ..., j)} = \frac{u_k(j, j, s_{-ik})}{u_k(j, j, ..., j)}.$$

If players have chosen the same strategy j, their payoffs may be different but the relative payoffs recalculated to a maximal-congestion profile $(j, j, ..., j)$ are equal. The profile $(j, j, ..., j)$ is characterized by maximal congestion or rivalry. In problems with positive or negative externalia, player payoffs for such profiles take the smallest or the greatest values, respectively. The relative values of player payoffs in the axiom (PS) demonstrate the equality of the players' shares in relation to the minimal or maximal payoff they can gain when using the strategy j.

Proportion of the maximal-congestion profile (PM). For any two different players $i, k \in N$ and for any two maximal-congestion profiles $s = (j, j, ..., j), s' = (j', j', ..., j') \in S^n$ such that $u_i(s') \neq 0, u_k(s') \neq 0$ we have

$$\frac{u_i(s)}{u_i(s')} = \frac{u_k(s)}{u_k(s')}.$$

For different maximal-congestion profiles the players' relative payoffs are equal. It is easy to see that the axioms (PS) and (PM) are a relaxation of (S) axiom, that is (S) $\Rightarrow$ (PS) and (S) $\Rightarrow$ (PM).

In the following, we show that the axioms (PS) and (PM) are involved in the characterization of WCGCs.

6.2 Characterization theorems

A characterization of singleton WCGCs in the cases $|N| \geq 2, |M| \geq 3$ is given in Theorem 10. The rest of the cases of $|N|$ and $|M|$ values are considered in Subsection 6.3.

Theorem 10. *Let $|N| \geq 2, |M| \geq 3, S = M$. The game $\Gamma \in G(N, M, S)$ satisfies the axioms (P), (PS), (PM), and (IIC) iff there exists arrays of positive players' constants and weights $\{\alpha_i\}_{i \in N}, \{w_i\}_{i \in N}$ and an array of resource functions $\{c_j\}_{j \in M}, c_j : W \to \mathbb{R}^+ \ \forall j \in M$, such that in the game Γ for $\forall i \in N \ \forall s \in S^n$ we have*

$$u_i(s) = \alpha_i \cdot c_{s_i}\left(\sum_{r \in K_{s_i}(s)} w_r\right).$$

Since WCGCs are WCGs, the characterization of singleton WCGCs contains the axioms that characterize singleton WCGs, i.e., (P) and (IIC). Theorem 10 makes it clear that to get a WCGC it is necessary and sufficient to require that the WCG satisfies the axioms (PS) and (PM).

A characterization of full WCGCs in the cases $|N| \geq 2, |M| \geq 2$ is given in Theorems 11 and 12. The rest of the cases of $|N|$ and $|M|$ values are considered in Subsection 6.3.

Theorem 11. *Let $|N| \geq 2, |M| \geq 2, S = 2^M$. The game $\Gamma \in G(N,M,S)$ satisfies the axioms (NN), (RA), (PS), and (PM) iff there exists arrays of positive player constants and weights $\{\alpha_i\}_{i\in N}, \{w_i\}_{i\in N}$ and an array of resource functions $\{c_j\}_{j\in M}, c_j : W \to \mathbb{R}^+ \ \forall j \in M$, such that in the game Γ for $\forall i \in N \ \forall s \in S^n$ we have*

$$u_i(s) = \alpha_i \cdot \sum_{j\in s_i} c_j \left(\sum_{r\in K_j(s)} w_r \right)$$

when $s_i \neq \emptyset$ and $u_i(s) = 0$ when $s_i = \emptyset$.

Theorem 12. *Let $|N| \geq 2, |M| \geq 2, S = 2^M$. The game $\Gamma \in G(N,M,S)$ satisfies the axioms (NN), (T), (PS), (PM), and (IIC) iff there exists arrays of positive player constants and weights $\{\alpha_i\}_{i\in N}, \{w_i\}_{i\in N}$ and an array of resource functions $\{c_j\}_{j\in M}, c_j : W \to \mathbb{R}^+ \ \forall j \in M$, such that in the game Γ for $\forall i \in N \ \forall s \in S^n$ we have*

$$u_i(s) = \alpha_i \cdot \sum_{j\in s_i} c_j \left(\sum_{r\in K_j(s)} w_r \right)$$

when $s_i \neq \emptyset$ and $u_i(s) = 0$ when $s_i = \emptyset$.

Proof. The proof is similar to the proof of Theorem 3. Theorem 12 follows from Theorem 11 and Lemma 4.

□

A conclusion from Theorems 10–12 is that a WCG is a WCGC iff the WCG satisfies the axioms (PS) and (PM).

6.3 Simple cases of WCGC characterization for small $|N|$ and $|M|$

Theorem 10 characterizes singleton WCGCs under the conditions $|N| \geq 2, |M| \geq 3$. Let us examine the $|N|$ and $|M|$ cases that are not covered by Theorem 10: 1. $|N| = 1$ or $|M| = 1$; 2. $|N| = 2, |M| = 2$; 3. $|N| \geq 3, |M| = 2$.

1. Let $|N| \geq 2, M = \{a\}$. In this case, there exists only one strategy profile $(a, a, ..., a) \in S^n$. We can represent the payoff of the player i as $u_i(a, a, ..., a) = \alpha_i \cdot c_a(w_1 + w_2 + ... + w_n)$, where $\alpha_i = u_i(a, a, ..., a), w_i = 1 \ \forall i \in N, c_a(n) = 1$. In this case, it is necessary and sufficient to satisfy the axiom (P).

Let $N = \{1\}, |M| \geq 1$. In this case, we can represent the payoff of player 1 as $u_1(s_1) = \alpha_1 \cdot c_{s_1}(w_1) \ \forall s_1 \in M$, where $\alpha_1 = 1, w_1 = 1, c_j(1) = u_1(j) \ \forall j \in M$. Similarly, satisfying the axiom (P) is necessary and sufficient here.

We can say that a WCGC, when $|N| = 1$ or $|M| = 1$, is characterized by the axiom (P).

2. Let $M = \{a, b\}$. We set the values of weights, constants, and resource functions in the same manner as in point 1 of the proof of Theorem 10, assuming that $x = a$. We get that $\alpha_1 = u_1(a, a), \alpha_2 = u_2(a, a)$ and, for example, $w_1 = 1, w_2 = 2$. The function $c_b(w_1 + w_2)$ takes

the values $\frac{u_1(b,b)}{u_1(a,a)}$ and $\frac{u_2(b,b)}{u_2(a,a)}$, their equality being warranted by the axiom (PM). In the other cases, the values of the resource functions are defined uniquely. In that event, it is necessary and sufficient to satisfy only the axioms (P) and (PM).

3. The values of weights, constants, and resource functions are set in the same manner as in the proof of Theorem 10. The difference is that for $|M| = 2$ the set $\overline{S}(K, j)$ consists of a single profile. Hence, the axiom (IIC) is not required for resource function uniqueness.

Theorem 11 characterizes full WCGCs under the conditions $|N| \geq 2, |M| \geq 2$. Let us examine the cases for the $|N|$ and $|M|$ that are not covered by Theorem 11: 1. $|N| = 1, |M| = 1$; 2. $|N| = 1, |M| \geq 2$; 3. $|N| \geq 2, |M| = 1$.

1. Let $N = \{1\}, M = \{a\}$, that is $|N| = |M| = 1$. Then, $S = 2^M = \{\emptyset, \{a\}\}$. We can represent the payoff of player 1 as $u_1(\emptyset) = 0$ and $u_1(\{a\}) = \alpha_1 \cdot c_a(w_1)$, where $\alpha_1 = 1, w_1 = 1, c_a(1) = u_1(\{a\})$. In that event, it is necessary and sufficient to satisfy the axiom (NN).

2. Let $N = \{1\}, |M| \geq 2, S = 2^M$. Here, the necessary and sufficient axioms for the existence of positive weights and positive-definite resource functions are (NN) and (RA). We find that $u_1(\{j\}) = \alpha_1 \cdot c_j(w_1)\ \forall j \in M$ and $u_1(s_1) = \alpha_1 \cdot \sum_{j \in s_1} c_j(w_1) = \sum_{j \in s_1} u_1(\{j\})\ \forall s_1 \in 2^M, s_1 \neq \emptyset$. Then, for example, $\alpha_1 = 1, w_1 = 1, c_j(1) = u_1(\{j\})\ \forall j \in M$.

3. Let $|N| \geq 2, M = \{a\}, S = 2^M = \{\emptyset, \{a\}\}$. We define player weights and the value of the resource function c_a in the same manner as in point 1 of the proof of Theorem 11, assuming that $x = a$. We get that $\alpha_i = u_i(\{a\}, \{a\}, ..., \{a\})\ \forall i \in N$. For the function c_a to be defined uniquely and to take positive values it is necessary and sufficient to satisfy the axioms (NN) and (PS). The axiom (PM) is unnecessary since there exists the only non-empty maximal-congestion profile $(\{a\}, \{a\}, ..., \{a\})$.

7. Conclusion

The results obtained on the characterization of WCGs help us determine the conditions under which WCGs can be applied to model problems with a congestion effect. It is enough to check a number of axioms to make sure that for players there exist weights and for resources there exist resource functions such that player payoffs in a normal-form game can be represented as player payoff functions in a WCG.

Acknowledgements

Support from the Basic Research Program of HSE University is gratefully acknowledged.

The author thanks Aleksei Kondratev, Alexander Nesterov, Alex Suzdaltsev, and the anonymous referee of HSE Working papers for very helpful comments.

Appendix

Proof of Theorem 1. It is not hard to check that a game with the specified players' payoff functions satisfies (P) and (IIC) axioms.

Let the game Γ satisfy (P) and (IIC). We will demonstrate that there exist positive weights of players and positive-definite resource functions such that each player's payoff has the specified form. Let us break the proof down into 3 points.

1. Definition of weights and resource functions. Let us set the positive numbers $w_1, w_2, ..., w_n$ so that $\sum_{r\in K_1} w_r \neq \sum_{r\in K_2} w_r \ \forall K_1, K_2 \subseteq N, K_1 \neq K_2$. In other words, all components of the vector $\left(\sum_{r\in K} w_r\right)_{\substack{K\subseteq N\\ K\neq\emptyset}}$ are pairwise different.

Let $K \subseteq N, K \neq \emptyset, j \in M$. We denote as $\overline{S}(K,j)$ a set that consists of the strategy profiles for which only players of the set K have chosen the resource j. Formally,

$$\overline{S}(K,j) = \{(s_1, s_2, ..., s_n) | s_i = j \ \forall i \in K \text{ and } s_i \neq j \ \forall i \in N \setminus K\}.$$

Since $|M| \geq 3$, the set $\overline{S}(K,j)$ consists of at least two strategy profiles.

Let us for each player $i \in N$, for each resource $j \in M$ and for each coalition $K \subseteq N, i \in K$ set the value of the resource function c_{ij} as follows,

$$c_{ij}\left(\sum_{r\in K} w_r\right) = u_i(\overline{s}(K,j)) \ \forall \overline{s}(K,j) \in \overline{S}(K,j).$$

It follows from (P) that the values of players' payoff functions are positive. Hence, the resource functions also take positive values.

2. Uniqueness of resource functions. Let us demonstrate that the function c_{ij} is defined uniquely for each pair $(i,j) \in N \times M$. Since all components of the vector $\left(\sum_{r\in K} w_r\right)_{\substack{K\subseteq N\\ K\neq\emptyset}}$ are pairwise different, we do not get a constraint on the resource function c_{ij} for the different coalitions K_1 and K_2, that is the numbers $c_{ij}\left(\sum_{r\in K_1} w_r\right)$ and $c_{ij}\left(\sum_{r\in K_2} w_r\right)$ for $K_1 \neq K_2$ can be either equal or not equal to each other.

For the function c_{ij} to be defined uniquely it is sufficient that for each coalition $K \subseteq N, K \neq \emptyset$ and for each $i \in K$ the following equality is satisfied:

$$u_i(\overline{s}(K,j)) = u_i(\overline{s}'(K,j)) \ \ \forall \overline{s}(K,j), \overline{s}'(K,j) \in \overline{S}(K,j).$$

Such an equality follows from (IIC). Indeed, according to the axiom (IIC), we can replace the strategies of players from the coalition $N \setminus K$ in the profile $\overline{s}'(K,j)$ with corresponding players' strategies in the profile $\overline{s}(K,j)$ without inducing a change in the payoff of each player in the coalition K. Hence, the resource function c_{ij} for each pair $(i,j) \in N \times M$ is defined uniquely.

3. Representation of players' payoffs. We have defined the values of players' weights and the values of resource functions. We can represent the payoff $u_i(s)$ as $u_i(\overline{s}(K_{s_i}(s), s_i))$, where

$s = \overline{s}(K_{s_i}(s), s_i) \in \overline{S}(K_{s_i}(s), s_i)$. Then, for $\forall i \in N$ $\forall s \in S^n$ we have

$$u_i(s) = u_i(\overline{s}(K_{s_i}(s), s_i)) = c_{is_i}\left(\sum_{r \in K_{s_i}(s)} w_r\right),$$

Q.E.D. $\square$

Proof of Theorem 2. It is not hard to check that a game with the specified players' payoff functions satisfies (NN) and (RA) axioms.

Let Γ satisfy (NN) and (RA). We will demonstrate that there exist positive players' weights and positive-definite resource functions such that the players' payoffs have the specified form. Let us break the proof down into 3 points.

1. Definition of weights and resource functions. For $K \subseteq N, K \neq \emptyset, j \in M$ we denote as $\overline{s}(K, j)$ the strategy profile in which players of the coalition K have chosen the strategy $\{j\}$, and the rest of the players have not chosen any resource, that is chose the $\emptyset$ strategy. Formally, $\overline{s}(K, j) = (\overline{s}_1(K, j), \overline{s}_2(K, j), ..., \overline{s}_n(K, j))$, where $\forall i \in N$:

$$\overline{s}_i(K, j) = \begin{cases} \{j\}, & i \in K; \\ \emptyset, & i \notin K. \end{cases}$$

Let us set the positive numbers $w_1, w_2, ..., w_n$ so that $\sum_{r \in K_1} w_r \neq \sum_{r \in K_2} w_r$ $\forall K_1, K_2 \subseteq N, K_1 \neq K_2$. In other words, all components of the vector $\left(\sum_{r \in K} w_r\right)_{\substack{K \subseteq N \\ K \neq \emptyset}}$ are pairwise different.

For $\forall i \in N$ $\forall K \subseteq N$ such that $i \in K$ and $\forall j \in M$ we define the values of the resource function c_{ij} as follows:

$$c_{ij}\left(\sum_{r \in K} w_r\right) = u_i(\overline{s}(K, j)).$$

2. Uniqueness of resource functions. Let us demonstrate that for each pair $(i, j) \in N \times M$ the function c_{ij} is defined uniquely. Since all components of the vector $\left(\sum_{r \in K} w_r\right)_{\substack{K \subseteq N \\ K \neq \emptyset}}$ are pairwise different, we do not get a constraint on the resource function c_{ij} for the different coalitions K_1 and K_2. Hence, the resource functions are defined uniquely.

It follows from (NN) that $u_i(\overline{s}(K, j)) > 0$ $\forall K \subseteq N, K \neq \emptyset, \forall i \in K, \forall j \in M$. Hence, the resource functions take positive values.

3. Representation of players' payoffs. We have defined the values of players' weights and the values of resource functions. Let us demonstrate that the players' payoff functions can be represented in the specified form.

Let $s = (s_i, s_{-i}) \in S^n$. If $s_i = \emptyset$, then it follows from (NN) that $u_i(s) = 0$. If $s_i \neq \emptyset$, then using point 2 of Lemma 1 we can transform $u_i(s)$ $\forall i \in N$ $\forall s \in S^n$ as follows:

$$u_i(s) = \sum_{j \in s_i} u_i(s_1 \cap \{j\}, s_2 \cap \{j\}, ..., s_n \cap \{j\}) = \sum_{j \in s_i} u_i\big(\overline{s}(K_j(s), j)\big) = \sum_{j \in s_i} c_{ij}\left(\sum_{r \in K_j(s)} w_r\right),$$

Q.E.D. $\square$

Proof of Lemma 1. 1. Let $u_i(s)$ have the specified form. Let's show that the game Γ satisfies (RA). We denote $s = (s_k, s_{-k}) \in S^n, s' = (s'_k, s'_{-k}) \in S^n, s'' = (s''_k, s''_{-k}) \in S^n$, where $s_k = s'_k \cup s''_k \ \forall k \in N$ and $(\cup_{k=1}^n s'_k) \cap (\cup_{k=1}^n s''_k) = \emptyset$. Then,

$$u_i(s) = \sum_{j \in \underset{k\in N}{\cup} s_k} u_i(s_1 \cap \{j\}, s_2 \cap \{j\}, ..., s_n \cap \{j\})$$

$$= \sum_{j \in \underset{k\in N}{\cup} s'_k} u_i(s'_1 \cap \{j\}, s'_2 \cap \{j\}, ..., s'_n \cap \{j\}) + \sum_{j \in \underset{k\in N}{\cup} s''_k} u_i(s''_1 \cap \{j\}, s''_2 \cap \{j\}, ..., s''_n \cap \{j\}) = u_i(s') + u_i(s'').$$

Therefore, Γ satisfies (RA).

Let Γ satisfy (RA). Let's show that the payoff $u_i(s)$ has the specified form. For the profile $s \in S^n$, we can separate the selected resources individually. Let $\cup_{k\in N} s_k = \{j, j', ...\}$, then

$$u_i(s) = u_i(s_1 \cap \{j\}, s_2 \cap \{j\}, ..., s_n \cap \{j\}) + u_i(s_1 \setminus \{j\}, s_2 \setminus \{j\}, ..., s_n \setminus \{j\})$$

$$= u_i(s_1 \cap \{j\}, s_2 \cap \{j\}, ..., s_n \cap \{j\}) + u_i(s_1 \cap \{j'\}, s_2 \cap \{j'\}, ..., s_n \cap \{j'\})$$

$$+ u_i(s_1 \setminus \{j, j'\}, s_2 \setminus \{j, j'\}, ..., s_n \setminus \{j, j'\}) = ... = \sum_{j \in \underset{k\in N}{\cup} s_k} u_i(s_1 \cap \{j\}, s_2 \cap \{j\}, ..., s_n \cap \{j\}).$$

2. Since Γ satisfies (NN), then $u_i(\emptyset, s_{-i}) = 0$. Therefore, $u_i(s_i \cap \{j\}, s_{-i}) = u_i(\emptyset, s_{-i}) = 0$, where $j \notin s_i$. Therefore, since the game Γ satisfies (RA), using the first point of the lemma, we can write down and simplify the player's winnings i as follows,

$$u_i(s) = \sum_{j \in \underset{k\in N}{\cup} s_k} u_i(s_1 \cap \{j\}, s_2 \cap \{j\}, ..., s_n \cap \{j\}) = \sum_{j \in s_i} u_i(s_1 \cap \{j\}, s_2 \cap \{j\}, ..., s_n \cap \{j\})$$

$$+ \sum_{j \in \underset{k\in N}{\cup} s_k \setminus s_i} u_i(s_1 \cap \{j\}, ..., s_{i-1} \cap \{j\}, \emptyset, s_{i+1} \cap \{j\}, ..., s_n \cap \{j\}) = \sum_{j \in s_i} u_i(s_1 \cap \{j\}, s_2 \cap \{j\}, ..., s_n \cap \{j\}),$$

Q. E. D. $\square$

Proof of Lemma 2. If $s_i = \emptyset$, then it follows from (NN) that $u_i(s) = 0$. Let $s_i = \{j_1, j_2, ..., j_{|s_i|}\}$. According to axiom (T), we have

$$u_i(\{j_1\}, s_{-i}) + u_i(\{j_2, j_3, ..., j_{|s_i|}\}, s_{-i}) = u_i(s_i, s_{-i}) + u_i(\{j_1\} \cap \{j_2, j_3, ..., j_{|s_i|}\}, s_{-i}).$$

Since Γ satisfies (NN), then $u_i(\{j_1\} \cap \{j_2, j_3, ..., j_{|s_i|}\}, s_{-i}) = u_i(\emptyset, s_{-i}) = 0$ and we can represent the above written equality as follows,

$$u_i(s_i, s_{-i}) = u_i(\{j_1\}, s_{-i}) + u_i(\{j_2, j_3, ..., j_{|s_i|}\}, s_{-i}).$$

Applying similar reasoning to $u_i(\{j_2, j_3, ..., j_{|s_i|}\}, s_{-i})$, we can separate resources individually,

$$u_i(s_i, s_{-i}) = u_i(\{j_1\}, s_{-i}) + u_i(\{j_2, j_3, ..., j_{|s_i|}\}, s_{-i})$$

$$= u_i(\{j_1\}, s_{-i}) + u_i(\{j_2\}, s_{-i}) + u_i(\{j_3, ..., j_{|s_i|}\}, s_{-i}) = ... = \sum_{j \in s_i} u_i(\{j\}, s_{-i}),$$

Q. E. D. □

Proof of Lemma 3. Since Γ satisfies (T), then

$$u_i(\{j\}, \{j\}, s_{-ik}) + u_i(\{j\}, s_k \setminus \{j\}, s_{-ik}) = u_i(\{j\}, s_k, s_{-ik}) + u_i(\{j\}, (s_k \setminus \{j\}) \cap \{j\}, s_{-ik})$$

$$\Leftrightarrow u_i(\{j\}, \{j\}, s_{-ik}) + u_i(\{j\}, s_k \setminus \{j\}, s_{-ik}) = u_i(\{j\}, s_k, s_{-ik}) + u_i(\{j\}, \emptyset, s_{-ik}).$$

Since Γ satisfies (IIC) and since $j \notin s_k \setminus \{j\}, j \notin \emptyset$, then $u_i(\{j\}, \emptyset, s_{-ik}) = u_i(\{j\}, s_k \setminus \{j\}, s_{-ik})$. Therefore, reducing the identical terms $u_i(\{j\}, \emptyset, s_{-ik})$ and $u_i(\{j\}, s_k \setminus \{j\}, s_{-ik})$ in the above equation, we get the equality $u_i(\{j\}, \{j\}, s_{-ik}) = u_i(\{j\}, s_k, s_{-ik})$,

Q.E.D. □

Proof of Lemma 4. Since Γ satisfies (NN) and (T), then by Lemma 2 we can represent the payoff of player i as

$$u_i(s) = u_i(s_i, s_{-i}) = \sum_{j \in s_i} u_i(\{j\}, s_{-i}) \; \forall s \in S^n.$$

Let s_k be the strategy of the player $k \neq i$ in the profile of s. If $j \notin s_k$, then according to axiom (II), the player's winnings i will not change if we replace the strategy s_k with $\emptyset$, since $\{j\} \cap s_k = \emptyset, \{j\} \cap \emptyset = \emptyset$. Then we can provide $u_i(\{j\}, s_{-i})$ as

$$u_i(\{j\}, s_{-i}) = u_i(\{j\}, \tilde{s}_{-i}),$$

where $\tilde{s} = (\{j\}, \tilde{s}_{-i})$ is a strategy profile whose components have the form

$$\tilde{s}_k = \begin{cases} s_k, & s_k \cap \{j\} = \{j\}, \\ \emptyset, & s_k \cap \{j\} = \emptyset. \end{cases} \quad \forall k \in N.$$

Let $K_j(\tilde{s}) = \{i\}$. Then in the profile $\tilde{s}$ all players except player i have chosen the strategy $\emptyset$ and we can write down the equality

$$u_i(\{j\}, s_{-i}) = u_i(\{j\}, \tilde{s}_{-i}) = u_i(s_1 \cap \{j\}, s_2 \cap \{j\}, ..., s_n \cap \{j\}).$$

Let $K_j(\tilde{s}) \neq \{i\}$. Then $\forall k \in K_j(\tilde{s}), k \neq i$ by Lemma 3 we have the equality

$$u_i(\{j\}, \tilde{s}_k, \tilde{s}_{-ik}) = u_i(\{j\}, \{j\}, \tilde{s}_{-ik}).$$

Therefore, for all players from the set $K_j(\tilde{s})$ in the profile $\tilde{s}$, we can delete resources that are not equal to j and the payoff of player i will not change, that is

$$u_i(\{j\}, s_{-i}) = u_i(\{j\}, \tilde{s}_{-i}) = u_i(s_1 \cap \{j\}, s_2 \cap \{j\}, ..., s_n \cap \{j\}).$$

Substituting the values $u_i(\{j\}, s_{-i}) = u_i(s_1 \cap \{j\}, s_2 \cap \{j\}, ..., s_n \cap \{j\})$ into the equality $u_i(s_i, s_{-i}) = \sum_{j \in s_i} u_i(\{j\}, s_{-i})$, we get

$$u_i(s) = \sum_{j \in s_i} u_i(s_1 \cap \{j\}, s_2 \cap \{j\}, ..., s_n \cap \{j\}).$$

It follows from (NN) that $u_i(s_i \cap \{j\}, s_{-i}) = 0$, if $j \notin s_i$. Therefore, we can add zero terms to the right side of the above equality and get equality

$$u_i(s) = \sum_{j \in \bigcup\limits_{k \in N} s_k} u_i(s_1 \cap \{j\}, s_2 \cap \{j\}, ..., s_n \cap \{j\}) \; \forall i \in N \; \forall s \in S^n.$$

Then, it follows from the first point of Lemma 1 that Γ satisfies (RA).

Q. E. D.

□

Proof of Lemma 5. 1. If $s_i = s_k = \emptyset$, then it follows from (NN) that $u_i(s) = 0 = u_k(s)$. If $s_i = s_k = \{j\}$, then the equality

$$u_i(\{j\}, \{j\}, s_{-ik}) - u_i(\emptyset, \{j\}, s_{-ik}) = u_k(\{j\}, \{j\}, s_{-ik}) - u_k(\{j\}, \emptyset, s_{-ik})$$

follows from (RMC). From (NN) we have $u_i(\emptyset, \{j\}, s_{-ik}) = 0 = u_k(\{j\}, \emptyset, s_{-ik})$. Removing the zero terms in the above equation for the axiom (RMC), we get that $u_i(s) = u_k(s)$.

2. Consider various arbitrary players $i, k \in N$ and an arbitrary profile $s \in S^n$, for which $s_i = s_k$. Since $S = M \cup \{\emptyset\}$,then $s_i = s_k = \emptyset$ or $s_i = s_k = j$, where $j \in M$. Since Γ satisfies (NN) and (RMC), then according to the first here we have $u_i(s) = u_k(s)$. Since equality holds for any various different players i, k who have chosen the same strategies in an arbitrary profile s, then (S) holds,

Q. E. D.

□

Proof of Theorem 4. It is not hard to check that a game with the specified players' payoff functions satisfies (P), (S) and (IIC) axioms.

Let the game Γ satisfy (P), (S) and (IIC). We will demonstrate that there exist positive players' weights and positive-definite resource functions such that each player's payoff has the specified form. Let us break the proof down into 3 points.

1. Definition of weights and resource functions. We set the players' weights and define the set $\overline{S}(K, j)$ in the same manner as in point 1 of the proof of Theorem 1.

Let us set $\forall K \subseteq N, K \neq \emptyset \; \forall j \in M$ the values of resource functions,

$$c_j\left(\sum_{r\in K} w_r\right) = u_i(\overline{s}(K,j)), \forall i \in K \; \forall \overline{s}(K,j) \in \overline{S}(K,j).$$

It follows from (P) that the values of players' payoff functions are positive. Hence, the resource functions also take positive values.

2. Uniqueness of resource functions. We will demonstrate that the function c_j is defined uniquely for each $j \in M$. Since all components of the vector $\left(\sum_{r\in K} w_r\right)_{\substack{K\subseteq N\\ K\neq\emptyset}}$ are pairwise different, we do not get a constraint on the resource function c_j for the different coalitions K_1 and K_2, that is the numbers $c_j\left(\sum_{r\in K_1} w_r\right)$ and $c_j\left(\sum_{r\in K_2} w_r\right)$ for $K_1 \neq K_2$ can be either equal or not equal to each other.

For the function c_j to be defined uniquely it is enough to satisfy the equality

$$u_i(\overline{s}(K,j)) = u_k(\overline{s}'(K,j)) \; \forall i,k \in K \; \forall \overline{s}(K,j), \overline{s}'(K,j) \in \overline{S}(K,j). \tag{1}$$

Let us demonstrate that this equality is satisfied. According to (IIC) axiom, we can replace the strategies of players from the coalition $N \setminus K$ in the profile $\overline{s}'(K,j)$ with the corresponding strategies of players in the profile $\overline{s}(K,j)$ without inducing a change in the payoff of each player of the coalition K, that is

$$u_i(\overline{s}'(K,j)) = u_i(\overline{s}(K,j)) \; \forall i \in K.$$

Since the players $i,k \in K$ have chosen the same strategy j in the profile $\overline{s}(K,j)$, (S) axiom warrants the equality $u_i(\overline{s}(K,j)) = u_k(\overline{s}(K,j))$. Hence, (1) is satisfied, wherefore for each $j \in M$ the function c_j is defined uniquely.

3. Representation of players' payoffs. We have defined the values of players' weights and resource functions. We can represent the payoff $u_i(s)$ as $u_i(\overline{s}(K_{s_i}(s), s_i))$, where $s = \overline{s}(K_{s_i}(s), s_i) \in \overline{S}(K_{s_i}(s), s_i)$. Then, $\forall i \in N \; \forall s \in S^n$ we have

$$u_i(s) = u_i(\overline{s}(K_{s_i}(s), s_i)) = c_{s_i}\left(\sum_{r\in K_{s_i}(s)} w_r\right),$$

Q.E.D. $\square$

Proof of Theorem 6. It is not hard to check that a game with the specified players' payoff functions satisfies (NN), (RA), and (S) axioms.

Let Γ satisfy (NN), (RA), and (S). We will demonstrate that there exist positive players' weights and positive-definite resource functions such that the players' payoffs have the specified form. Let us break the proof down into 3 points.

1. Definition of weights and resource functions. We set the players' weights and define the profile $\overline{s}(K,j)$ in the same manner as in point 1 of the proof of Theorem 2.

For $\forall K \subseteq N, K \neq \emptyset, \forall j \in M$ we define the values of the resource function c_j as follows:

$$c_j\left(\sum_{r\in K} w_r\right) = u_i(\overline{s}(K,j))\ \forall i \in K.$$

2. Uniqueness of resource functions. We will demonstrate that the function c_j is uniquely defined for each $j \in M$. Since all components of the vector $\left(\sum_{r\in K} w_r\right)_{\substack{K\subseteq N\\K\neq\emptyset}}$ are pairwise different, we do not get a constraint on the resource function c_j for the different coalitions K_1 and K_2. For the function c_j to be defined uniquely it is enough to satisfy the equality

$$u_i(\overline{s}(K,j)) = u_k(\overline{s}(K,j))\ \forall i,k \in K,$$

which follows from (S) axiom.

It follows from (NN) that $u_i(\overline{s}(K,j)) > 0\ \forall K \subseteq N, K \neq \emptyset, \forall i \in K, \forall j \in M$. Hence, the resource functions take positive values.

3. Representation of players' payoffs. We have defined the values of players' weights and the values of resource functions. Let us demonstrate that the players' payoff functions can be represented in the specified form.

Let $s = (s_i, s_{-i}) \in S^n$. If $s_i = \emptyset$, then it follows from (NN) that $u_i(s) = 0$. If $s_i \neq \emptyset$, then using point 2 of Lemma 1 we can transform $u_i(s)\ \forall i \in N\ \forall s \in S^n$ as follows:

$$u_i(s) = \sum_{j\in s_i} u_i(s_1 \cap \{j\}, s_2 \cap \{j\}, ..., s_n \cap \{j\}) = \sum_{j\in s_i} u_i\big(\overline{s}(K_j(s),j)\big) = \sum_{j\in s_i} c_j\left(\sum_{r\in K_j(s)} w_r\right),$$

Q.E.D.

□

Proof of Theorem 10. It is not hard to demonstrate that the specified players' payoff functions satisfy the listed axioms.

Let the game Γ satisfy (P), (PS), (PM), and (IIC). We will demonstrate that there exist arrays of positive constants and players' weights as well as an array of resource functions such that the payoffs of players in the game Γ can be represented in the specified form. The proof is broken down into three points.

1. Definition of weights, constants, and resource functions. We set the values of the weights $w_1, w_2, ..., w_n$ and define the set $\overline{S}(K,j)$ in the same manner as in point 1 of the proof of Theorem 1.

Let us fix an arbitrary resource $x \in M$. We set $c_x\left(\sum_{r\in N} w_r\right) = 1$. Then, $u_i(x,x,...,x) = \alpha_i \cdot c_x\left(\sum_{r\in N} w_r\right) = \alpha_i$. Hence, $\alpha_i = u_i(x,x,...,x)$. It follows from (P) that $u_i(x,x,...,x) > 0\ \forall i \in N$, wherefore $\alpha_i > 0\ \forall i \in N$. Let us set

$$c_j\left(\sum_{r\in K} w_r\right) = \frac{u_i(\overline{s}(K,j))}{u_i(x,x,...,x)}, \forall K \subseteq N\ \forall i \in K\ \forall \overline{s}(K,j) \in \overline{S}(K,j).$$

2. Uniqueness of resource functions. Let us demonstrate that for each $j \in M$ the resource function c_j is defined uniquely. Since all components of the vector $\left(\sum_{r\in K} w_r\right)_{\substack{K\subseteq N\\ K\neq\emptyset}}$ are pairwise different, we do not get a constraint on the resource function c_j for the different coalitions K_1 and K_2, because the values of the function c_j with different arguments can be either equal or not equal to each other.

For the function c_j to be defined uniquely it is enough to satisfy the proportion

$$\frac{u_i(\overline{s}(K,j))}{u_i(x,x,...,x)} = \frac{u_k(\overline{s}'(K,j))}{u_k(x,x,...,x)} \; \forall i,k \in K \; \forall \overline{s}(K,j), \overline{s}'(K,j) \in \overline{S}(K,j). \tag{2}$$

Let us demonstrate that this equality is satisfied. According to (IIC) axiom, we can replace the strategies of players from the coalition $N \setminus K$ in the profile $\overline{s}'(K,j)$ with the corresponding strategies of players in the profile $\overline{s}(K,j)$ without inducing a change in the payoff of the player i, that is

$$u_i(\overline{s}'(K,j)) = u_i(\overline{s}(K,j)) \; \forall i \in K.$$

In (2), we substitute $u_k(\overline{s}'(K,j))$ with the number $u_k(\overline{s}(K,j))$ and transform the proportion as follows,

$$\frac{u_i(\overline{s}(K,j))}{u_i(j,j,...,j)} \cdot \frac{u_i(j,j,...,j)}{u_i(x,x,...,x)} = \frac{u_k(\overline{s}(K,j))}{u_k(j,j,...,j)} \cdot \frac{u_k(j,j,...,j)}{u_k(x,x,...,x)}.$$

It follows from (PS) and (PM) that

$$\frac{u_i(\overline{s}(K,j))}{u_i(j,j,...,j)} = \frac{u_k(\overline{s}(K,j))}{u_k(j,j,...,j)} \text{ and } \frac{u_i(j,j,...,j)}{u_i(x,x,...,x)} = \frac{u_k(j,j,...,j)}{u_k(x,x,...,x)},$$

respectively. Hence, (2) is true. This means that the resource functions are defined uniquely. It follows from (P) that the values of players' payoff functions are positive, wherefore the resource functions also take positive values.

3. Representation of players' payoffs. We have defined the constants and players' weights as well as the values of resource functions. We can represent the payoff $u_i(s)$ as $u_i(\overline{s}(K_{s_i}(s), s_i))$, where $\overline{s}(K_{s_i}(s), s_i) \in \overline{S}(K_{s_i}(s), s_i)$ and use the equality $c_j\left(\sum_{r\in K} w_r\right) = \frac{u_i(\overline{s}(K,j))}{u_i(x,x,...,x)}$ to express $u_i(\overline{s}(K,j))$. Then, $\forall i \in N \; \forall s \in S^n$ we have

$$u_i(s) = u_i(\overline{s}(K_{s_i}(s), s_i)) = u_i(x,x,...,x) \cdot c_{s_i}\left(\sum_{r\in K_{s_i}(s)} w_r\right) = \alpha_i \cdot c_{s_i}\left(\sum_{r\in K_{s_i}(s)} w_r\right),$$

Q.E.D. □

Proof of Theorem 11. It is not hard to demonstrate that the specified payoff functions satisfy the listed axioms.

Let Γ satisfy (NN), (RA), (PS), and (PM). Let us demonstrate that there exist arrays of positive constants and players' weights as well as an array of resource functions such that the payoffs of players in the game Γ can be represented in the specified form. We will break the proof down into 3 points.

1. Definition of weights, constants, and resource functions. We set the weight values $w_1, w_2, ..., w_n$ and define the profile $\overline{s}(K, j)$ in the same manner as in point 1 of the proof of Theorem 2.

Let us fix an arbitrary resource $x \in M$. We set $c_x\left(\sum_{r\in N} w_r\right) = 1$. Then $u_i(x, x, ..., x) = \alpha_i \cdot c_x\left(\sum_{r\in N} w_r\right) = \alpha_i$. Hence, $\alpha_i = u_i(x, x, ..., x)$. It follows from (NN) that $u_i(x, x, ..., x) > 0\ \forall i \in N$, wherefore $\alpha_i > 0\ \forall i \in N$.

For $\forall K \subseteq N, K \neq \emptyset, \forall j \in M$ we define the values of the function c_j as follows,

$$c_j\left(\sum_{r\in K} w_r\right) = \frac{u_i(\overline{s}(K, j))}{u_i(x, x, ..., x)},\ \forall i \in K.$$

2. Uniqueness of resource functions. Let us demonstrate that the function c_j for each $j \in M$ is defined uniquely. For that to be true it is enough that the following equality is satisfied

$$\frac{u_i(\overline{s}(K, j))}{u_i(x, x, ..., x)} = \frac{u_k(\overline{s}(K, j))}{u_k(x, x, ..., x)}\ \forall i, k \in K. \tag{3}$$

Let us demonstrate that (3) is satisfied. We rewrite the above proportion in the form

$$\frac{u_i(\overline{s}(K, j))}{u_i(j, j, ..., j)} \cdot \frac{u_i(j, j, ..., j)}{u_i(x, x, ..., x)} = \frac{u_k(\overline{s}(K, j))}{u_k(j, j, ..., j)} \cdot \frac{u_k(j, j, ..., j)}{u_k(x, x, ..., x)} \forall i, k \in K.$$

It follows from (PS) and (PM) that

$$\frac{u_i(\overline{s}(K, j))}{u_i(j, j, ..., j)} = \frac{u_k(\overline{s}(K, j))}{u_k(j, j, ..., j)} \text{ and } \frac{u_i(j, j, ..., j)}{u_i(x, x, ..., x)} = \frac{u_k(j, j, ..., j)}{u_k(x, x, ..., x)},$$

respectively. Hence, (3) is true, that is the function c_j is defined uniquely for each $j \in M$. Considering the explicit form of the resource functions, it follows from (NN) that the resource functions take positive values.

3. Representation of players' payoffs. We have defined the constants and players' weights and the values of resource functions. Let us demonstrate that the players' payoff functions can be represented in the specified form.

Let $s = (s_i, s_{-i}) \in S^n$. If $s_i = \emptyset$, then it follows from (NN) that $u_i(s) = 0$. If $s_i \neq \emptyset$, then considering the equality

$$u_i(\overline{s}(j, K)) = u_i(x, x, ..., x) \cdot c_j\left(\sum_{r\in K} w_r\right) = \alpha_i \cdot c_j\left(\sum_{r\in K} w_r\right)\ \forall i \in N\ \forall j \in M\ \forall K \subseteq N, K \neq \emptyset$$

and using point 2 of Lemma 1 we can transform $u_i(s)\ \forall i \in N\ \forall s \in S^n$ as follows,

$$u_i(s) = \sum_{j\in s_i} u_i(s_1 \cap \{j\}, s_2 \cap \{j\}, ..., s_n \cap \{j\}) = \sum_{j\in s_i} u_i\big(\overline{s}(j, K_j(s))\big)$$

$$= \sum_{j \in s_i} \alpha_i \cdot c_j \left(\sum_{r \in K_j(s)} w_r \right) = \alpha_i \cdot \sum_{j \in s_i} c_j \left(\sum_{r \in K_j(s)} w_r \right),$$

Q.E.D.

□

References

Besner, M. (2019). Axiomatizations of the proportional Shapley value. *Theory and Decision*, 86(2), 161–183.

Bilò, V. (2007). On satisfiability games and the power of congestion games. *In: Kao, MY., Li, XY. (eds) Algorithmic Aspects in Information and Management. AAIM 2007. Lecture Notes in Computer Science, vol 4508. Springer, Berlin, Heidelberg.*, 231–240.

Bilò, V., Gourvès, L., & Monnot, J. (2023). Project games. *Theoretical Computer Science*, 940, 97–111.

Bogomolnaia, A., & Jackson, M. O. (2002). The stability of hedonic coalition structures. *Games and Economic Behavior*, 38(2), 201–230.

Chun, Y. (1989). A new axiomatization of the Shapley value. *Games and Economic Behavior*, 1(2), 119–130.

Dreze, J. H., & Greenberg, J. (1980). Hedonic coalitions: Optimality and stability. *Econometrica: Journal of the Econometric Society*, 48(4), 987–1003.

Dubey, P., Einy, E., & Haimanko, O. (2005). Compound voting and the Banzhaf index. *Games and Economic Behavior*, 51(1), 20–30.

Einy, E., & Haimanko, O. (2011). Characterization of the Shapley–Shubik power index without the efficiency axiom. *Games and Economic Behavior*, 73(2), 615–621.

Gómez-Rúa, M., & Vidal-Puga, J. (2010). The axiomatic approach to three values in games with coalition structure. *European Journal of Operational Research*, 207(2), 795–806.

Gusev, V., Nesterov, A., Reshetov, M., & Suzdaltsev, A. (2024). The existence of a pure-strategy Nash equilibrium in a discrete ponds dilemma. *Games and Economic Behavior*, 147, 38–51.

Hollard, G. (2000). On the existence of a pure strategy Nash equilibrium in group formation games. *Economics Letters*, 66(3), 283–287.

Kalai, E., & Samet, D. (1987). On weighted Shapley values. *International Journal of Game Theory*, 16(3), 205–222.

Konishi, H., Le Breton, M., & Weber, S. (1997a). Equilibria in a model with partial rivalry. *Journal of Economic Theory*, 72(1), 225–237.

Konishi, H., Le Breton, M., & Weber, S. (1997b). Equivalence of strong and coalition-proof Nash equilibria in games without spillovers. *Economic Theory*, 9, 97–113.

Mavronicolas, M., Milchtaich, I., Monien, B., & Tiemann, K. (2007). Congestion games with player-specific constants. *Mathematical Foundations of Computer Science 2007: 32nd*

International Symposium, MFCS 2007 Český Krumlov, Czech Republic, August 26-31, 2007 Proceedings 32, 633–644.

Milchtaich, I. (1996). Congestion games with player-specific payoff functions. *Games and Economic Behavior*, 13(1), 111–124.

Milchtaich, I. (2009). Weighted congestion games with separable preferences. *Games and Economic Behavior*, 67(2), 750–757.

Milchtaich, I. (2013). Representation of finite games as network congestion games. *International Journal of Game Theory*, 42, 1085–1096.

Milchtaich, I. (2021). Internalization of social cost in congestion games. *Economic Theory*, 71(2), 717–760.

Monderer, D. (2007). Multipotential games. *Sangal R, Mehta H, Bagga RK, eds.Proc. 20nd Internat. Joint Conf. Artificial Intelligence (Morgan Kaufmann, San Francisco)*, 1422–1427.

Monderer, D., & Shapley, L. S. (1996). Potential games. *Games and Economic Behavior*, 14(1), 124–143.

Myerson, R. B. (1977). Graphs and cooperation in games. *Mathematics of Operations Research*, 2(3), 225–229.

Nowak, A. S., & Radzik, T. (1995). On axiomatizations of the weighted Shapley values. *Games and Economic Behavior*, 8(2), 389–405.

Rosenthal, R. W. (1973). A class of games possessing pure-strategy Nash equilibria. *International Journal of Game Theory*, 2, 65–67.

Shapley, L. S. (1971). Cores of convex games. *International Journal of Game Theory*, 1, 11–26.

Thomson, W. (2001). On the axiomatic method and its recent applications to game theory and resource allocation. *Social Choice and Welfare*, 18(2), 327–386.

Ui, T. (2000). A Shapley value representation of potential games. *Games and Economic Behavior*, 31(1), 121–135.

van den Brink, R., & Pinter, M. (2015). On axiomatizations of the Shapley value for assignment games. *Journal of Mathematical Economics*, 60, 110–114.

Young, H. P. (1985). Monotonic solutions of cooperative games. *International Journal of Game Theory*, 14(2), 65–72.

Zou, Z., van den Brink, R., Chun, Y., & Funaki, Y. (2021). Axiomatizations of the proportional division value. *Social Choice and Welfare*, 57, 35–62.

Vasily Gusev

National Research University Higher School of Economics (Saint Petersburg, Russia). International Laboratory of Game Theory and Decision Making. Senior Research Fellow.

E-mail: vgusev@hse.ru

Any opinions or claims contained in this Working Paper do not necessarily reflect the views of HSE.

© Gusev 2025